Worksheet 1

Differences

Remember:
The **difference** of a sequence u_n is:

$$\Delta u_n = u_{n+1} - u_n$$

The **falling factorial (power)**, n to the k falling, when $k > 0$, is:

$$n^{\underline{k}} = n(n-1)\dots(n-k+1),$$
$$n^{\underline{0}} = 1,$$
$$n^{\underline{-k}} = \frac{1}{(n+1)(n+2)\dots(n+k)}.$$

Please complete the following table:

Sequence u_n	Difference Δu_n
c	
n	
n^2	
$n^{\underline{2}}$	
$n^{\underline{k}}$	
$n^{\underline{-1}}$	
$n^{\underline{-k}}$	
c^n	

Worksheet 2

Indefinite Sums

Remember:
To evaluate the sum:

$$\sum_n u_n$$

we search for a sequence U such that $u_n = \Delta U_n$.

The **falling factorial (power)**, n to the k falling, when $k > 0$, is:

$$n^{\underline{k}} = n(n-1)\dots(n-k+1),$$
$$n^{\underline{0}} = 1,$$
$$n^{\underline{-k}} = \frac{1}{(n+1)(n+2)\dots(n+k)}.$$

Please complete the following table:

Sequence u_n	Sum $\sum_n u_n$
c	
n	
$n^{\underline{2}}$	
$n^{\underline{k}}$	
$n^{\underline{-1}}$	
$n^{\underline{-k}}$	
c^n	

Worksheet 3

Definite Sums

Remember the fundamental theorem of sum calculus (the method of differences): If $u_r = \Delta U_r$ then

$$\sum_{r=a}^{b} u_r = U_r \Big|_a^{b+1} = U_{b+1} - U_a$$

Evaluate the following sums using the fundamental theorem of sum calculus. When possible use the table of indefinite sums of Worksheet 2.

1.

$$\sum_{r=0}^{n}(2r+1)$$

2.

$$\sum_{r=a}^{b} c^r$$

3.

$$\sum_{r=a}^{b} \binom{r}{k-1}$$

Hint: Define

$$U_r = \binom{r}{k}$$

and use Pascal's rule:

$$\binom{r}{k-1} + \binom{r}{k} = \binom{r+1}{k}$$

4.

$$\sum_{r=0}^{n} r^{\underline{1}}$$

5.

$$\sum_{r=0}^{n} r^{\underline{2}}$$

6.

$$\sum_{r=0}^{n} r^{\underline{k}}$$

Worksheet 4

Sums of powers

Remember, we used:

$$\sum_{r=0}^{n} r^{\underline{k}} = \frac{(n+1)^{\underline{k+1}}}{k+1}$$

to compute the sum of squares this way:

$$\sum_{r=0}^{n} r^2 = \sum_{r=0}^{n} (r^{\underline{2}} + r^{\underline{1}}) = \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}$$

$$= \frac{(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} = \frac{n(n+1)(2n+1)}{6}$$

1. Using this technique compute the sum of cubes:

$$\sum_{r=0}^{n} r^3 = ?$$

Hint: start writing $r^{\underline{3}}$ in terms of powers and then write r^3 in terms of falling factorial powers.

Worksheet 5

Sum by parts

Remember, the sum by parts formula for definite sums is:

$$\sum_{r=a}^{b} u_r \Delta v_r = u_{b+1} v_{b+1} - u_a v_a - \sum_{r=a}^{b} v_{r+1} \Delta u_r$$

1. Evaluate this sum by parts:

$$\sum_{r=0}^{n} r 2^r$$

Worksheet 6

Linear difference equations with constant coefficients

Remember: Assume $u_n = m^n$ and $m \neq 0$. If the auxiliary equation has distinct roots α and β then the general solution is:

$$u_n = A\alpha^n + B\beta^n.$$

If the roots are the same, α, then the general solution is:

$$u_n = (An + B)\alpha^n.$$

1. Find the general solution of this difference equation:

$$u_{n+2} = 6u_{n+1} - 8u_n$$

2. Solve this difference equation:

$$u_{n+2} = 8u_{n+1} - 15u_n, \qquad u_0 = 2, \quad u_1 = 8$$

3. Find a closed formula for this constant-recursive sequence:

$$u_{n+2} = 4u_{n+1} - 4u_n, \qquad u_0 = 2, \quad u_1 = 6$$

4. Solve this recurrence relation:

$$u_{n+2} = u_{n+1} + u_n, \qquad u_0 = 0, \quad u_1 = 1$$

Answers - Worksheet 1
Differences

Remember: The **difference** of a sequence u_n is:

$$\Delta u_n = u_{n+1} - u_n$$

The **falling factorial (power)**, n to the k falling, when $k > 0$, is:

$$n^{\underline{k}} = n(n-1)\dots(n-k+1),$$
$$n^{\underline{0}} = 1,$$
$$n^{\underline{-k}} = \frac{1}{(n+1)(n+2)\dots(n+k)}.$$

Please complete the following table:

Sequence u_n	Difference Δu_n
c	0
n	1
n^2	$2n+1$
$n^{\underline{2}}$	$2n$
$n^{\underline{k}}$	$kn^{\underline{k-1}}$
$n^{\underline{-1}}$	$-n^{\underline{-2}}$
$n^{\underline{-k}}$	$-kn^{\underline{-k-1}}$
c^n	$c^n(c-1)$

Answers - Worksheet 2
Indefinite Sums

Remember, to evaluate the sum:

$$\sum_n u_n$$

we search for a sequence U such that $u_n = \Delta U_n$.

Please complete the following table:

Sequence u_n	Sum $\sum_n u_n$
c	cn
n	$\frac{n^{\underline{2}}}{2}$
$n^{\underline{2}}$	$\frac{n^{\underline{3}}}{3}$
$n^{\underline{k}}$	$\frac{n^{\underline{k+1}}}{k+1}$
$n^{\underline{-1}}$	**Harmonic number** $H_n = \sum_{r=1}^{n} \frac{1}{r}$, the discrete version of $ln(x) = \int_1^x \frac{1}{r}\mathrm{d}r$
$n^{\underline{-k}}$	$\frac{n^{\underline{-k+1}}}{-k+1}$
c^n	$\frac{c^n}{c-1}$

Answers - Worksheet 3
Definite sums

Remember the fundamental theorem of sum calculus (the method of differences): If $u_r = \Delta U_r$ then

$$\sum_{r=a}^{b} u_r = U_r \Big|_a^{b+1} = U_{b+1} - U_a$$

Evaluate the following sums using the fundamental theorem of sum calculus. When possible use the table of indefinite sums of Worksheet 2.

1.

$$\sum_{r=0}^{n} (2r+1)$$

We need to find U_r such that $\Delta U_r = 2r + 1$:

$$U_r = r^2 \implies \Delta U_r = (r+1)^2 - r^2 = 2r + 1$$

$$\therefore \quad \sum_{r=0}^{n} (2r+1) = r^2 \Big|_0^{n+1} = (n+1)^2 - 0^2 = (n+1)^2$$

2.

$$\sum_{r=a}^{b} c^r$$

$$U_r = \frac{c^r}{c-1} \implies \Delta U_r = \frac{c^{r+1}}{c-1} - \frac{c^r}{c-1} = c^r$$

$$\therefore \quad \sum_{r=a}^{b} c^r = \frac{c^r}{c-1} \Bigg|_a^{b+1} = \frac{c^{b+1} - c^a}{c-1}$$

3.

$$\sum_{r=a}^{b} \binom{r}{k-1}$$

$$U_r = \binom{r}{k}$$

$$\Delta U_r = \binom{r+1}{k} - \binom{r}{k} = \binom{r}{k-1} \qquad \text{(Pascal's rule)}$$

$$\therefore \sum_{r=a}^{b} \binom{r}{k-1} = \binom{r}{k} \Bigg|_a^{b+1} = \binom{b+1}{k} - \binom{a}{k}$$

4.

$$\sum_{r=0}^{n} r^{\underline{1}}$$

$$\begin{aligned} U_r = \frac{r^{\underline{2}}}{2} &\implies \Delta U_r = r^{\underline{1}} \\ \therefore \quad \sum_{r=0}^{n} r^{\underline{1}} &= \left.\frac{r^{\underline{2}}}{2}\right|_{0}^{n+1} \\ &= \frac{(n+1)^{\underline{2}}}{2} - \frac{0^{\underline{2}}}{2} \\ &= \frac{(n+1)^{\underline{2}}}{2} \end{aligned}$$

5.

$$\sum_{r=0}^{n} r^{\underline{2}}$$

$$\begin{aligned} U_r = \frac{r^{\underline{3}}}{3} &\implies \Delta U_r = r^{\underline{2}} \\ \therefore \quad \sum_{r=0}^{n} r^{\underline{2}} &= \left.\frac{r^{\underline{3}}}{3}\right|_{0}^{n+1} \\ &= \frac{(n+1)^{\underline{3}}}{3} - \frac{0^{\underline{3}}}{3} \\ &= \frac{(n+1)^{\underline{3}}}{3} \end{aligned}$$

6.

$$\sum_{r=0}^{n} r^{\underline{k}}$$

$$\begin{aligned} U_r = \frac{r^{\underline{k+1}}}{k+1} &\implies \Delta U_r = r^{\underline{k}} \\ \therefore \quad \sum_{r=0}^{n} r^{\underline{k}} &= \left.\frac{r^{\underline{k+1}}}{k+1}\right|_{0}^{n+1} \\ &= \frac{(n+1)^{\underline{k+1}}}{k+1} - \frac{0^{\underline{k+1}}}{k+1} \\ &= \frac{(n+1)^{\underline{k+1}}}{k+1} \end{aligned}$$

Answers - Worksheet 4

Sums of powers

Remember, we used:

$$\sum_{r=0}^{n} r^{\underline{k}} = \frac{(n+1)^{\underline{k+1}}}{k+1}$$

to compute the sum of squares this way:

$$\sum_{r=0}^{n} r^2 = \sum_{r=0}^{n} (r^{\underline{2}} + r^{\underline{1}}) = \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}$$

$$= \frac{(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} = \frac{n(n+1)(2n+1)}{6}$$

1. Using this technique compute the sum of cubes:

$$\sum_{r=0}^{n} r^3 = ?$$

   Hint: start writing $r^{\underline{3}}$ in terms of powers and then write r^3 in terms of falling factorial powers.

$$\begin{aligned} r^{\underline{3}} &= r(r-1)(r-2) = r^3 - 3r^2 + 2r \\ r^3 &= r^{\underline{3}} + 3r^2 - 2r = r^{\underline{3}} + 3(r^{\underline{2}} + r^{\underline{1}}) - 2r^{\underline{1}} \\ \therefore r^3 &= r^{\underline{3}} + 3r^{\underline{2}} + r^{\underline{1}} \end{aligned}$$

$$\begin{aligned} \therefore \sum_{r=0}^{n} r^3 &= \sum_{r=0}^{n} (r^{\underline{3}} + 3r^{\underline{2}} + r^{\underline{1}}) = \frac{(n+1)^{\underline{4}}}{4} + \frac{3(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2} \\ &= \frac{(n+1)n(n-1)(n-2)}{4} + \frac{3(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} \\ &= (n+1)n\left(\frac{n^2-3n+2}{4} + \frac{4n-4}{4} + \frac{2}{4}\right) = (n+1)n\left(\frac{n^2+n}{4}\right) \\ &= \frac{n^2(n+1)^2}{4} \end{aligned}$$

Worksheet 5
Sum by parts

Remember, the sum by parts formula for definite sums is:

$$\sum_{r=a}^{b} u_r \Delta v_r = u_{b+1} v_{b+1} - u_a v_a - \sum_{r=a}^{b} v_{r+1} \Delta u_r$$

1. Evaluate this sum by parts:

$$\sum_{r=0}^{n} r2^r$$

We let $u_r = r$ and $\Delta v_r = 2^r$, then $\Delta u_r = 1$, $v_r = 2^r$ and:

$$\begin{aligned}\sum_{r=0}^{n} r2^r &= (n+1)2^{n+1} - 0 \cdot 2^0 - \sum_{r=0}^{n} 2^{r+1} \cdot 1 \\ &= (n+1)2^{n+1} - 2\sum_{r=0}^{n} 2^r \\ &= (n+1)2^{n+1} - 2(2^{n+1} - 1) \\ &= (n-1)2^{n+1} + 2\end{aligned}$$

Answers - Worksheet 6
Linear difference equations with constant coefficients

1. Find the general solution of this difference equation:

$$u_{n+2} = 6u_{n+1} - 8u_n$$

Assuming $u_n = m^n$ and $n \neq 0$, we get:

$$\begin{aligned}
&m^{n+2} = 6m^{n+1} - 8m^n \\
&m^2 - 6m + 8 = 0 && \text{(Auxiliary equation)} \\
&(m-2)(m-4) = 0 \\
&\therefore \alpha = 2, \beta = 4 \\
&\therefore u_n = A\,2^n + B\,4^n && \text{(General solution)}
\end{aligned}$$

2. Solve this difference equation:

$$u_{n+2} = 8u_{n+1} - 15u_n, \qquad u_0 = 2, \quad u_1 = 8$$

Assuming $u_n = m^n$ and $n \neq 0$, we get:

$$\begin{aligned}
&m^{n+2} = 8m^{n+1} - 15m^n \\
&m^2 - 8m + 15 = 0 && \text{(Auxiliary equation)} \\
&(m-3)(m-5) = 0 \\
&\therefore \alpha = 3, \beta = 5 \\
&\therefore u_n = A\,3^n + B\,5^n && \text{(General solution)} \\
&A + B = 2 && (u_0 = 2) \\
&3A + 5B = 8 && (u_1 = 8) \\
&\therefore A = 1, B = 1 \\
&\therefore u_n = 3^n + 5^n
\end{aligned}$$

3. Find a closed formula for this constant-recursive sequence:

$$u_{n+2} = 4u_{n+1} - 4u_n, \qquad u_0 = 2, \quad u_1 = 6$$

$$\begin{aligned}
&m^2 - 4m + 4 = 0 && \text{(Auxiliary equation)} \\
&(m-2)^2 = 0 \\
&\therefore \alpha = 2 && \text{(Repeated root)} \\
&\therefore u_n = (An + B)2^n && \text{(General solution)} \\
&(A0 + B)2^0 = 2 && (u_0 = 2) \\
&(A1 + B)2^1 = 6 && (u_1 = 6) \\
&\therefore A = 1, B = 2 \\
&\therefore u_n = (n+2)2^n
\end{aligned}$$

4. Solve this recurrence relation:

$$u_{n+2} = u_{n+1} + u_n, \qquad u_0 = 0, \quad u_1 = 1$$

Assuming $u_n = m^n$ and $n \neq 0$, we get:

$$m^{n+2} = m^{n+1} + m^n$$

$$m^2 = m + 1 \qquad \text{(Auxiliary equation)}$$

$$\therefore \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2} \qquad \text{(Golden ratio and its conjugate)}$$

$$\therefore u_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n \qquad \text{(General solution)}$$

$$A + B = 0 \qquad (u_0 = 0)$$

$$A\frac{1+\sqrt{5}}{2} + B\frac{1-\sqrt{5}}{2} = 1 \qquad (u_1 = 1)$$

$$A - B = \frac{2}{\sqrt{5}}$$

$$\therefore A = \frac{1}{\sqrt{5}}, B = -\frac{1}{\sqrt{5}}$$

$$\therefore u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This closed-form expression for the Fibonacci sequence is known as Binet's formula.