Discrete calculus

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Discrete calculus

The purpose of this presentation is to give an introduction to a version of calculus that applies to sequences. It is known as finite or discrete calculus.

This relatively unknown theory has, surprisingly, many analogous results and parallel techniques to those of traditional calculus.

This presentation is aimed at UK Year 13 students that are taking Further Maths A-level.

Discrete calculus

The basic objects of study in traditional calculus are functions that have real numbers as both input and output (domain and range):

$$f : \mathbb{R} \to \mathbb{R}$$

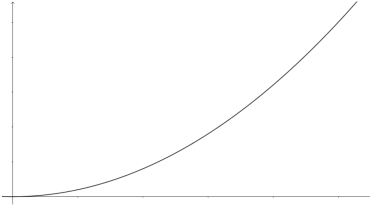
In discrete calculus the functions have as input the natural numbers:

$$u : \mathbb{N} \to \mathbb{R}$$

We traditionally call these functions sequences and we use the notation u_n instead of $u(n)$. We use n instead of x to emphasize that n is a natural number.

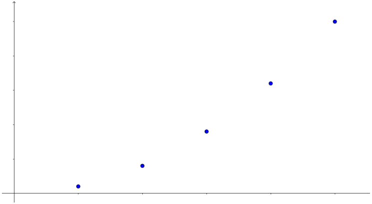
Discrete calculus

Continuous function:



$$f(x) = x^2$$

Discrete function (sequence):



$$u_n = n^2$$

Derivative and difference

Let's recall the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In the discrete case the smallest possible h is 1:

$$\frac{f(x+1) - f(x)}{1}$$

This leads to the discrete version of the derivative. The **difference** of a sequence u_n is defined as:

$$\Delta u_n = u_{n+1} - u_n$$

Difference

The difference of a sequence u_n is:

$$\Delta u_n = u_{n+1} - u_n$$

Examples:

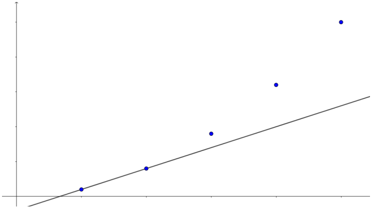
n	0	1	2	3	4	5	Sequence type
u_n	7	7	7	7	7	7	Constant
Δu_n	0	0	0	0	0		Zero
u_n	0	6	12	18	24	30	Linear
Δu_n	6	6	6	6	6		Constant
u_n	0	1	4	9	16	25	Quadratic
Δu_n	1	3	5	7	9		Linear
u_n	1	2	4	8	16	32	Geometric (Exponential)
Δu_n	1	2	4	8	16		Geometric (Exponential)

The differences behave similarly to the derivatives.

Geometric interpretation of the difference

The difference of a sequence u_n is:

$$\Delta u_n = u_{n+1} - u_n$$



$$u_n = n^2$$

$\Delta u_1 = u_2 - u_1 = 4 - 1 = 3$ is the gradient (slope) of the line that goes through $(1, u_1)$ and $(2, u_2)$.

In general, Δu_n is the gradient of the line that goes through (n, u_n) and $(n+1, u_{n+1})$.

This is analogous to: $f'(x)$ is the gradient of the tangent of $y = f(x)$ that goes through $(x, f(x))$.

Difference of n^2

The difference of $u_n = n^2$ is:

$$\Delta u_n = (n+1)^2 - n^2 = 2n + 1$$

We would like to obtain a similar result to

$$f(x) = x^2 \implies f'(x) = 2x$$

For that purpose we define the **falling factorial (power)**:

$$n^{\underline{2}} = n(n-1)$$

Then:

$$\begin{aligned} u_n = n^{\underline{2}} \implies \Delta u_n &= (n+1)^{\underline{2}} - n^{\underline{2}} \\ &= (n+1)n - n(n-1) \\ &= 2n \end{aligned}$$

Worksheet 1

The **difference** of a sequence u_n is:

$$\Delta u_n = u_{n+1} - u_n$$

In general, the **falling factorial (power)**, n to the k falling, when $k > 0$, is:

$$n^{\underline{k}} = n(n-1)\dots(n-k+1)$$
$$n^{\underline{0}} = 1$$
$$n^{\underline{-k}} = \frac{1}{(n+1)(n+2)\dots(n+k)}$$

Difference of a falling factorial

The **falling factorial (power)**, n to the k falling when $k > 0$, is:

$$n^{\underline{k}} = n(n-1)\dots(n-k+1)$$

If $u_n = n^{\underline{k}}$ then:

$$\begin{aligned}
\Delta u_n &= (n+1)^{\underline{k}} - n^{\underline{k}} \\
&= \Big((n+1)n\dots(n-k+2)\Big) - \Big(n(n-1)\dots(n-k+1)\Big) \\
&= n(n-1)\dots(n-k+2)\Big(n+1-(n-k+1)\Big) \\
&= kn^{\underline{k-1}}
\end{aligned}$$

Therefore:

$$u_n = n^{\underline{k}} \implies \Delta u_n = kn^{\underline{k-1}}$$

This is the discrete version of:

$$f(x) = x^k \implies f'(x) = kx^{k-1}$$

Difference of a falling factorial

When the exponent is negative the definition is:

$$n^{\underline{-k}} = \frac{1}{(n+1)(n+2)\dots(n+k)}$$

If $u_n = n^{\underline{-k}}$ then:

$$\begin{aligned}
\Delta u_n &= (n+1)^{\underline{-k}} - n^{\underline{-k}} \\
&= \frac{1}{(n+2)(n+3)\dots(n+k+1)} - \frac{1}{(n+1)(n+2)\dots(n+k)} \\
&= \frac{n+1-(n+k+1)}{(n+1)(n+2)\dots(n+k+1)} \\
&= \frac{-k}{(n+1)(n+2)\dots(n+k+1)} \\
&= -kn^{\underline{-k-1}}
\end{aligned}$$

$$\therefore u_n = n^{\underline{-k}} \implies \Delta u_n = -kn^{\underline{-k-1}}$$

Difference of a geometric (exponential) sequence

If u_n is a geometric sequence we have:

$$u_n = c^n \implies \Delta u_n = c^{n+1} - c^n = c^n(c-1)$$

In particular if $c = 2$:

$$u_n = 2^n \implies \Delta u_n = 2^n$$

This is the discrete version of the exponential function:

$$f(x) = e^x \implies f'(x) = e^x$$

Another way to see that 2 is the discrete version of e is:

$$2^n = (1+1)^n = \sum_{r=0}^{n} \binom{n}{r} = \sum_{r=0}^{n} \frac{n(n-1)\ldots(n-r+1)}{r!} = \sum_{r=0}^{\infty} \frac{n^{\underline{r}}}{r!}$$

This is similar to the exponential function's MacLaurin series:

$$e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

Properties of differences

If u_n and v_n are sequences and c is a constant:

$$\Delta(u_n + v_n) = \Delta u_n + \Delta v_n \qquad \text{(Linearity 1)}$$

$$\Delta(cu_n) = c\Delta u_n \qquad \text{(Linearity 2)}$$

$$\begin{aligned}\Delta(u_n v_n) &= (\Delta u_n)v_{n+1} + u_n \Delta v_n \\ &= (\Delta u_n)v_n + u_n \Delta v_n + \Delta u_n \Delta v_n \qquad \text{(Product rule)}\end{aligned}$$

$$\Delta\left(\frac{u_n}{v_n}\right) = \frac{(\Delta u_n)v_n - u_n \Delta v_n}{v_n v_{n+1}} \qquad \text{(Quotient rule)}$$

These are similar to:

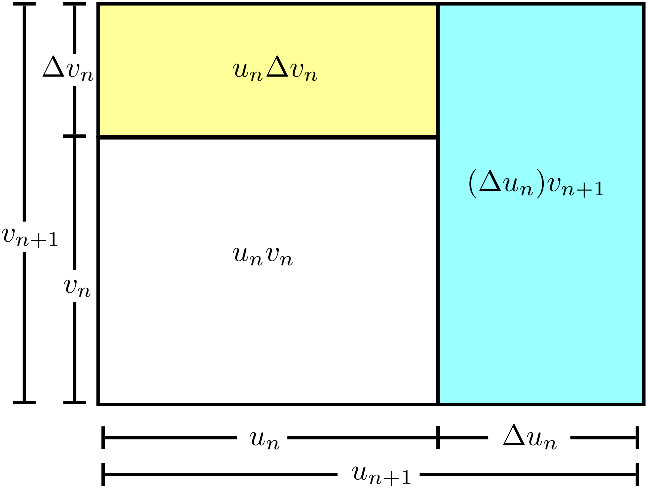
$$(f+g)'(x) = f'(x) + g'(x) \qquad \text{(Linearity 1)}$$

$$(cf)'(x) = cf'(x) \qquad \text{(Linearity 2)}$$

$$(f.g)'(x) = f'(x)g(x) + f(x)g'(x) \qquad \text{(Product rule)}$$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \qquad \text{(Quotient rule)}$$

Geometric proof of the product rule



$$\Delta(u_n v_n) = u_{n+1} v_{n+1} - u_n v_n = (\Delta u_n) v_{n+1} + u_n \Delta v_n$$

Indefinite sum

The indefinite integral, also known as the antiderivative, is the inverse operation to the derivative. That is, to evaluate the integral:

$$\int f(x)\mathrm{d}x$$

we search for a function F such that $f(x) = F'(x)$.

The discrete version is the indefinite sum, also known as the antidifference. It is the inverse operation to the difference. To evaluate the sum:

$$\sum_{n} u_n$$

we search for a sequence U such that $u_n = \Delta U_n$.

Indefinite sum

Another way to describe the indefinite integral is:

$$\int f'(x)\mathrm{d}x = f(x) + C$$

and the analogous for the indefinite sum is:

$$\sum_n \Delta u_n = u_n + C$$

Leibniz introduced the integral symbol based on the long letter S (a letter that fell out of use in the 19th century) because he thought of the integral as an infinite sum of infinitesimal summands. He even used the name 'calculus summatorius', the name integral calculus appeared later.

It was common to use S for summation until Euler's capital sigma, the Greek S, became popular.

Worksheet 2

To evaluate the sum:

$$\sum_n u_n$$

we search for a sequence U such that $u_n = \Delta U_n$.

Elementary differences and sums

Sequence	Difference	Sum
u_n	Δu_n	$\sum\limits_n u_n$
c	0	cn
n	1	$\frac{n^{\underline{2}}}{2}$
$n^{\underline{k}}$	$kn^{\underline{k-1}}$	$\frac{n^{\underline{k+1}}}{k+1}, k \neq -1$
c^n	$c^n(c-1)$	$\frac{c^n}{c-1}$
$n^{\underline{-1}}$	$-n^{\underline{-2}}$	**Harmonic number** $H_n = \sum_{r=1}^{n} \frac{1}{r}$, the discrete version of $ln(x) = \int_1^x \frac{1}{r} \mathrm{d}r$

Properties of indefinite sums

If u_n and v_n are sequences and c is a constant:

$$\sum (u_n + v_n) = \sum u_n + \sum v_n \qquad \text{(Linearity 1)}$$

$$\sum (c u_n) = c \sum u_n \qquad \text{(Linearity 2)}$$

$$\sum u_n \Delta v_n = u_n v_n - \sum v_{n+1} \Delta u_n \qquad \text{(Sum by parts)}$$

These are similar to:

$$\int [f + g](x) \mathrm{d}x = \int f(x) \mathrm{d}x + \int g(x) \mathrm{d}x \qquad \text{(Linearity 1)}$$

$$\int c f(x) \mathrm{d}x = c \int f(x) \mathrm{d}x \qquad \text{(Linearity 2)}$$

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x \qquad \text{(Integration by parts)}$$

Note that the sum and integration by parts formulas come from the respective product rules.

Definite sum

The fundamental theorem of calculus: If $f(x) = F'(x)$ then

$$\int_a^b f(x)\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a)$$

has this discrete version: If $u_r = \Delta U_r$ then

$$\sum_{r=a}^{b} u_r = U_r\Big|_a^{b+1} = U_{b+1} - U_a$$

This is the method of differences to compute telescoping sums:

$$\begin{aligned}\sum_{r=a}^{b} u_r &= \sum_{r=a}^{b} \Delta U_r \\ &= (U_{a+1} - U_a) + (U_{a+2} - U_{a+1}) + \cdots + (U_{b+1} - U_b) \\ &= U_{b+1} - U_a\end{aligned}$$

Fundamental theorem of sum calculus

If $u_r = \Delta U_r$ then

$$\sum_{r=a}^{b} u_r = U_r \Big|_a^{b+1} = U_{b+1} - U_a$$

Let's solve a classic example of the method of differences using the fundamental theorem of sum calculus:

$$\sum_{r=a}^{b} \frac{1}{r(r+1)}$$

$$U_r = -\frac{1}{r} \implies \Delta U_r = -\frac{1}{r+1} - \left(-\frac{1}{r}\right) = \frac{1}{r(r+1)}$$

$$\therefore \sum_{r=a}^{b} \frac{1}{r(r+1)} = -\frac{1}{r}\Big|_a^{b+1} = -\frac{1}{b+1} - \left(-\frac{1}{a}\right) = -\frac{1}{b+1} + \frac{1}{a}$$

Worksheet 3

Fundamental theorem of sum calculus: If $u_r = \Delta U_r$ then

$$\sum_{r=a}^{b} u_r = U_r\Big|_a^{b+1} = U_{b+1} - U_a$$

Fundamental theorem of sum calculus

If $u_r = \Delta U_r$ then

$$\sum_{r=a}^{b} u_r = U_r \Big|_a^{b+1} = U_{b+1} - U_a$$

$$\sum_{r=0}^{n} (2r+1) = ?$$

To evaluate this using the fundamental theorem of sum calculus, we need to find U_r such that $\Delta U_r = 2r + 1$:

$$U_r = r^2 \implies \Delta U_r = (r+1)^2 - r^2 = 2r + 1$$

$$\therefore \quad \sum_{r=0}^{n} (2r+1) = r^2 \Big|_0^{n+1} = (n+1)^2 - 0^2 = (n+1)^2$$

Fundamental theorem of sum calculus

If $u_r = \Delta U_r$ then

$$\sum_{r=a}^{b} u_r = U_r \Big|_a^{b+1} = U_{b+1} - U_a$$

$$\sum_{r=a}^{b} c^r = \, ?$$

$$U_r = \frac{c^r}{c-1} \implies \Delta U_r = \frac{c^{r+1}}{c-1} - \frac{c^r}{c-1} = c^r$$

$$\therefore \quad \sum_{r=a}^{b} c^r = \frac{c^r}{c-1} \Bigg|_a^{b+1} = \frac{c^{b+1} - c^a}{c-1}$$

In particular, $$\sum_{r=0}^{b} c^r = \frac{c^{b+1} - 1}{c-1}$$ (Geometric series)

Fundamental theorem of sum calculus

If $u_r = \Delta U_r$ then

$$\sum_{r=a}^{b} u_r = U_r\Big|_a^{b+1} = U_{b+1} - U_a$$

$$\sum_{r=a}^{b} \binom{r}{k-1} = ?$$

$$U_r = \binom{r}{k}$$

$$\Delta U_r = \binom{r+1}{k} - \binom{r}{k} = \binom{r}{k-1} \quad \text{(Pascal's rule)}$$

$$\therefore \sum_{r=a}^{b} \binom{r}{k-1} = \binom{r}{k}\Bigg|_a^{b+1} = \binom{b+1}{k} - \binom{a}{k}$$

Fundamental theorem of sum calculus

If $u_r = \Delta U_r$ then

$$\sum_{r=a}^{b} u_r = U_r \Big|_{a}^{b+1} = U_{b+1} - U_a$$

$$\sum_{r=0}^{n} r^{\underline{k}} = \, ?$$

$$U_r = \frac{r^{\underline{k+1}}}{k+1} \implies \Delta U_r = r^{\underline{k}}$$

$$\therefore \quad \sum_{r=0}^{n} r^{\underline{k}} = \frac{r^{\underline{k+1}}}{k+1} \Bigg|_{0}^{n+1} = \frac{(n+1)^{\underline{k+1}}}{k+1}$$

The continuous version of this is:

$$\int_0^n x^k \mathrm{d}x = \frac{n^{k+1}}{k+1}$$

Sums of powers

We can use

$$\sum_{r=0}^{n} r^{\underline{k}} = \frac{(n+1)^{\underline{k+1}}}{k+1}$$

to compute sums of powers:

$$\sum_{r=0}^{n} r = \sum_{r=0}^{n} r^{\underline{1}} = \frac{(n+1)^{\underline{2}}}{2}$$

$$= \frac{(n+1)n}{2}$$

Sums of powers

$$\sum_{r=0}^{n} r^{\underline{k}} = \frac{(n+1)^{\underline{k+1}}}{k+1}$$

What about the sum of squares?

$$\begin{aligned}
r^{\underline{2}} &= r(r-1) = r^2 - r \\
\therefore r^2 &= r^{\underline{2}} + r^{\underline{1}} \\
\sum_{r=0}^{n} r^2 = \sum_{r=0}^{n} (r^{\underline{2}} + r^{\underline{1}}) &= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2} \\
&= \frac{(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Sums of powers

Worksheet 4

$$\sum_{r=0}^{n} r^{\underline{k}} = \frac{(n+1)^{\underline{k+1}}}{k+1}$$

$$\sum_{r=0}^{n} r^2 = \sum_{r=0}^{n} (r^{\underline{2}} + r^{\underline{1}}) = \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}$$

Using this technique compute the sum of cubes:

$$\sum_{r=0}^{n} r^3 = ?$$

Sums of powers

$$\sum_{r=0}^{n} r^{\underline{k}} = \frac{(n+1)^{\underline{k+1}}}{k+1}$$

$$\begin{aligned}
r^{\underline{3}} &= r(r-1)(r-2) = r^3 - 3r^2 + 2r \\
r^3 &= r^{\underline{3}} + 3r^2 - 2r = r^{\underline{3}} + 3(r^{\underline{2}} + r^{\underline{1}}) - 2r^{\underline{1}} \\
\therefore r^3 &= r^{\underline{3}} + 3r^{\underline{2}} + r^{\underline{1}} \\
\sum_{r=0}^{n} r^3 &= \sum_{r=0}^{n} (r^{\underline{3}} + 3r^{\underline{2}} + r^{\underline{1}}) = \frac{(n+1)^{\underline{4}}}{4} + \frac{3(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2} \\
&= \frac{(n+1)n(n-1)(n-2)}{4} + \frac{3(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} \\
&= \frac{n^2(n+1)^2}{4}
\end{aligned}$$

Sum by parts

The sum by parts formula for definite sums is:

$$\sum_{r=a}^{b} u_r \Delta v_r = u_{b+1} v_{b+1} - u_a v_a - \sum_{r=a}^{b} v_{r+1} \Delta u_r$$

For example, to evaluate this sum by parts:

$$\sum_{r=0}^{n} 2r3^r$$

we let $u_r = r$ and $\Delta v_r = 2 \cdot 3^r$, then $\Delta u_r = 1$, $v_r = 3^r$ and:

$$\sum_{r=0}^{n} 2r3^r = (n+1)3^{n+1} - 0 \cdot 3^0 - \sum_{r=0}^{n} 3^{r+1} \cdot 1$$

$$= (n+1)3^{n+1} - 3\sum_{r=0}^{n} 3^r = (n+1)3^{n+1} - 3\frac{3^{n+1}-1}{3-1}$$

$$= 3^{n+1}\left(n - \frac{1}{2}\right) + \frac{3}{2}$$

Worksheet 5

The sum by parts formula for definite sums is:

$$\sum_{r=a}^{b} u_r \Delta v_r = u_{b+1} v_{b+1} - u_a v_a - \sum_{r=a}^{b} v_{r+1} \Delta u_r$$

Exercise: evaluate this sum by parts:

$$\sum_{r=0}^{n} r 2^r$$

Sum by parts

The sum by parts formula for definite sums is:

$$\sum_{r=a}^{b} u_r \Delta v_r = u_{b+1} v_{b+1} - u_a v_a - \sum_{r=a}^{b} v_{r+1} \Delta u_r$$

To evaluate this sum by parts:

$$\sum_{r=0}^{n} r 2^r$$

we let $u_r = r$ and $\Delta v_r = 2^r$, then $\Delta u_r = 1$, $v_r = 2^r$ and:

$$\begin{aligned}\sum_{r=0}^{n} r 2^r &= (n+1)2^{n+1} - 0 \cdot 2^0 - \sum_{r=0}^{n} 2^{r+1} \cdot 1 \\ &= (n+1)2^{n+1} - 2\sum_{r=0}^{n} 2^r \\ &= (n+1)2^{n+1} - 2(2^{n+1} - 1) \\ &= (n-1)2^{n+1} + 2\end{aligned}$$

Linear differential and difference equations

In this context the analogous to the derivative is the **shift operator**: $Eu_n = u_{n+1}$. The method to solve linear equations is extremely similar:

Differential equation	Difference equation
$y'' - 5y' + 6y = 0$	$u_{n+2} - 5u_{n+1} + 6u_n = 0$
Assume $y = e^{mx}$	Assume $u_n = m^n$ and $m \neq 0$
then $y' = me^{mx}$	then $u_{n+1} = m^{n+1}$
and $y'' = m^2 e^{mx}$	and $u_{n+2} = m^{n+2}$
$\therefore m^2 e^{mx} - 5me^{mx} + 6e^{mx} = 0$	$\therefore m^{n+2} - 5m^{n+1} + 6m^n = 0$
$\therefore$ the auxiliary equation is	$\therefore$ the auxiliary equation is
$m^2 - 5m + 6 = 0$	$m^2 - 5m + 6 = 0$
$(m-2)(m-3) = 0$	$(m-2)(m-3) = 0$
$\therefore$ the general solution is	$\therefore$ the general solution is
$y = Ae^{2x} + Be^{3x}$	$u_n = A \cdot 2^n + B \cdot 3^n$

where A and B are arbitrary constants.

Linear differential and difference equations with constant coefficients

Vector space

$$y = A\, e^{2x} + B\, e^{3x} \qquad u_n = A\, 2^n + B\, 3^n \qquad \mathbf{v} = A\, \mathbf{i} + B\, \mathbf{j}$$

Vector spaces are the subject of linear algebra.

Linear differential and difference equations

In both cases the general solution depends on the roots of the auxiliary equation:

AE's roots	Differential equation	Difference equation
	Assume $y = e^{mx}$	Assume $u_n = m^n$ and $m \neq 0$
Distinct ($\mathbb{R}$ or $\mathbb{C}$) $\alpha \neq \beta$	$y = Ae^{\alpha x} + Be^{\beta x}$	$u_n = A\alpha^n + B\beta^n$
Equal α	$y = (Ax + B)e^{\alpha x}$	$u_n = (An + B)\alpha^n$
Complex $p \pm iq$ $\rho e^{\pm i\theta}$	$y = e^{px}(A\cos qx + B\sin qx)$	$u_n = \rho^n(A\cos n\theta + B\sin n\theta)$

Linear differential and difference equations

If the equation is not homogeneous:

general solution = complimentary function $+$ particular integral.

The boundary conditions for difference equations are usually some initial values.

In Further Maths you study the case of order 2. This can be generalized to higher orders.

Difference equations are also called recurrence relations. A sequence satisfying a linear recurrence with constant coefficients is called a constant-recursive sequence.

Linear difference equations

Example: Find a formula for the periodic sequence:

$$11, 1, 3, 1, 11, 1, 3, 1, \ldots$$

$$u_{n+4} = u_n \quad \text{with } u_0 = 11, u_1 = 1, u_2 = 3, u_3 = 1$$

Assuming $u_n = m^n$ and $m \neq 0$, we get:

$$m^{n+4} = m^n$$

$$m^4 = 1 \qquad \text{(Auxiliary equation)}$$

$$\therefore m_1 = 1, m_2 = -1, m_3 = i, m_4 = -i \qquad \text{(Roots of unity)}$$

$$\therefore u_n = A \cdot 1^n + B(-1)^n + Ci^n + D(-i)^n \qquad \text{(General solution)}$$

$$A + B + C + D = 11 \qquad (u_0 = 11)$$

$$A - B + Ci - Di = 1 \qquad (u_1 = 1)$$

$$A + B - C - D = 3 \qquad (u_2 = 3)$$

$$A - B - Ci + Di = 1 \qquad (u_3 = 1)$$

$$\therefore u_n = 4 + 3(-1)^n + 2i^n + 2(-i)^n$$

Linear difference equations

Assume $u_n = m^n$ and $m \neq 0$. If the auxiliary equation has distinct roots α and β then the general solution is:

$$u_n = A\alpha^n + B\beta^n$$

Exercise: Using this technique and assuming that the first term is u_0, find a formula for the periodic sequence:

$$20, 10, 20, 10, 20, 10, \ldots$$

A very simple periodic sequence

Find a formula for the periodic sequence:

$$20, 10, 20, 10, 20, 10, \ldots$$

$$u_{n+2} = u_n \quad \text{with } u_0 = 20, u_1 = 10$$

Assuming $u_n = m^n$ and $n \neq 0$, we get $m^{n+2} = m^n$ and:

$$
\begin{aligned}
& m^2 = 1 && \text{(Auxiliary equation)} \\
& \therefore \alpha = 1, \beta = -1 && \text{(Roots of unity)} \\
& \therefore u_n = A \cdot 1^n + B(-1)^n = A + B(-1)^n && \text{(General solution)} \\
& A + B = 20 && (u_0 = 20) \\
& A - B = 10 && (u_1 = 10) \\
& \therefore A = 15, B = 5 && \\
& \therefore u_n = 15 + 5(-1)^n &&
\end{aligned}
$$

In general, if the period is n we would have the n-th roots of 1.

Linear differential and difference equations with constant coefficients

Worksheet 6

Assume $u_n = m^n$ and $m \neq 0$. If the auxiliary equation has distinct roots α and β then the general solution is:

$$u_n = A\alpha^n + B\beta^n.$$

If the roots are the same, α, then the general solution is:

$$u_n = (An + B)\alpha^n.$$

Fibonacci sequence

$$u_{n+2} = u_{n+1} + u_n$$
$$u_0 = 0, u_1 = 1$$

Assuming $u_n = m^n$ and $n \neq 0$, we get:

$$m^{n+2} = m^{n+1} + m^n$$

$$m^2 = m + 1 \qquad \text{(Auxiliary equation)}$$

$$\therefore \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2} \qquad \text{(Golden ratio and its conjugate)}$$

$$\therefore u_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n \qquad \text{(General solution)}$$

$$A + B = 0 \qquad (u_0 = 0)$$

$$A\frac{1+\sqrt{5}}{2} + B\frac{1-\sqrt{5}}{2} = 1, \quad A - B = \frac{2}{\sqrt{5}} \qquad (u_1 = 1)$$

$$\therefore u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This closed-form expression is known as Binet's formula.

Is it all a coincidence?

A more general definition of a difference of a function f is:

$$\Delta_h f(x) = f(x+h) - f(x)$$

where h is a positive number.
The difference that we have studied is the particular case when $h = 1$:

$$\Delta u_n = \Delta u(n) = \Delta_1 u(n)$$

Using Δ_h we can rewrite the definition of the derivative like this:

$$f'(x) = \lim_{h \to 0} \frac{\Delta_h f(x)}{h}$$

This equality explains why the properties of differences and derivatives are so similar. There is an analogous relationship between sums and integrals, just check the definition of Riemann integrals.

Applications and discrete mathematics

Numerical calculus: discretize differential equations to solve them numerically. For example, Euler's method consists of approximating a function by a sequence defined by a recurrence relation.

Computer science: recurrence relations are used to analyze recursive algorithms.

Many branches of mathematics have discrete and continuous versions: probability distributions, Fourier transforms and dynamical systems among many others.

The appearance of digital, as opposed to analog, computers in the 20^{th} century increased the importance of discrete mathematics.

Summary

Continuous calculus	Discrete calculus
Functions	Sequences
Derivative	Difference
x^k	$n^{\underline{k}}$
Integral	Sum
Fundamental theorem of calculus	Method of differences
Differential equations	Difference equations (Recurrence relations)
e^{mx}	m^n
Laplace transform	Generating function

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- Robert Ghrist, University of Pennsylvania. Discrete Calculus videos: Sequences I, Sequences II, Differences and Discrete Calculus.
- Richard Feynman, Feynman lectures. California Institute of Technology. Newtons Laws of Dynamics. It includes nice examples of discretization of differential equations.
- Wikipedia: Finite difference, Indefinite sum, Recurrence relation, Linear difference equation, Constant-recursive sequence.

- You can find this presentation at gustavolau.com.