Logic

Mathematics Masterclass

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Introduction

What is logic?

Logic: the study of valid reasoning.

Informal logic: the study of natural language arguments.

Mathematical logic, according to the mathematician and philosopher Bertrand Russell: "The subject in which nobody knows what one is talking about, nor whether what one is saying is true."

Introduction

More seriously, mathematical logic has many applications and it is fundamental in digital electronics and computer programming.

We are going to start from simple ideas but the tools that we are going to study allow to analyse complex logical problems. We will use these tools to solve some non-trivial logical puzzles.

Knights and liars

Video from *Labyrinth* movie, directed by Jim Henson:

https://www.youtube.com/watch?v=2dgmgub8mHw

Worksheet 1
Reasoning

Do you agree with the following arguments?

a) Pam promised she would attend the school assembly or send a substitute. She is not coming to the assembly, so we are expecting a substitute.

b) If there are delays in the Circle line then Jenny is late at work. Today she has arrived late therefore there are delays in the Circle line.

c) If there are delays in the Circle line then Jenny is late at work. Today she has arrived on time therefore there are no delays in the Circle line.

Logic algebra

In this algebra instead of numbers we will work with statements that can have two values: True (T) and False (F).

Usually this is called Boolean algebra in honour of the 19th century English mathematician George Boole.

Sometimes we use Yes/No or 1/0 instead of True/False.

We normally use the letters p, q, r, etc. to represent statements. For example p could mean “There are delays in the Circle line” or “Today Jenny was late at work”.

Boolean algebra

Most of the statements that we are going to use are about things or people having certain properties. Examples:

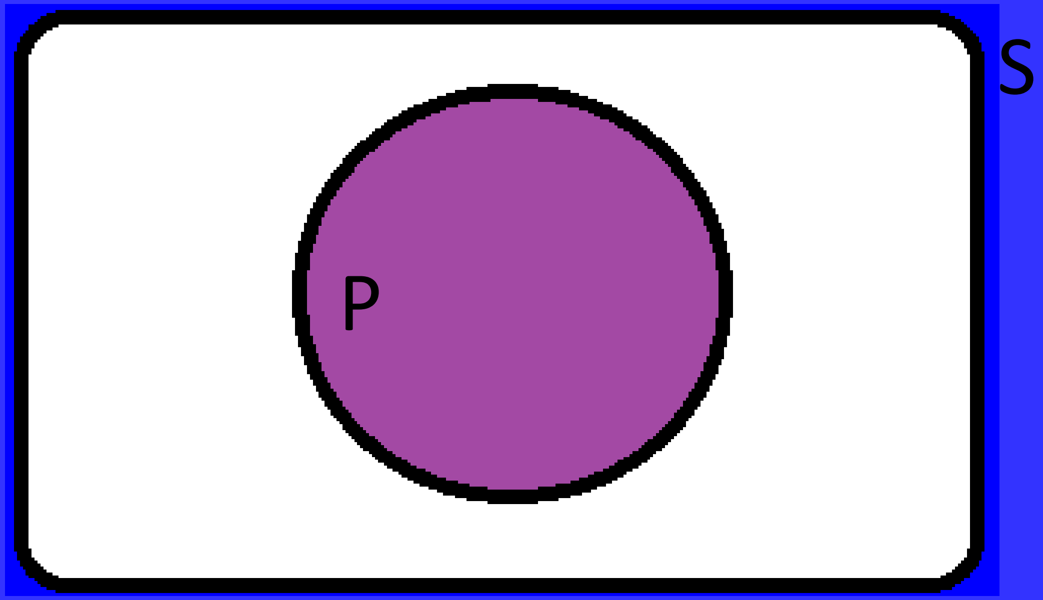
-The Circle line is not working: Circle line is the thing, if it’s working or not is a property.

-Jenny is late: Jenny is a person, being late is a property.

We can think of properties as sets. The set of underground lines that are working, the set of people that arrive late, etc.

Boolean algebra

Therefore an important case to consider is when p means “x is a member of the set P” which is abbreviated as $x \in P$.

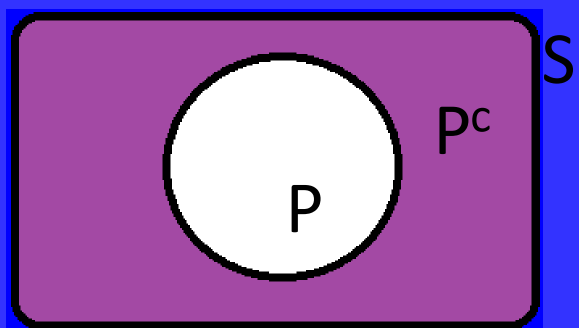


Example: The big box is the students in this class and P is the set of girls in this class.

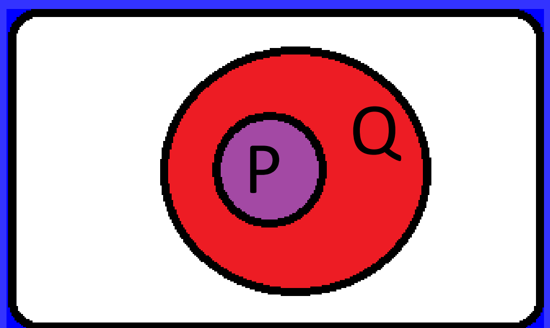
Set operations

We are going to be using sets operations and relations.

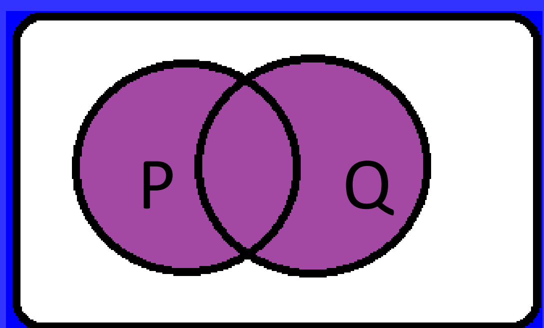
An easy way to understand them is with Venn diagrams:



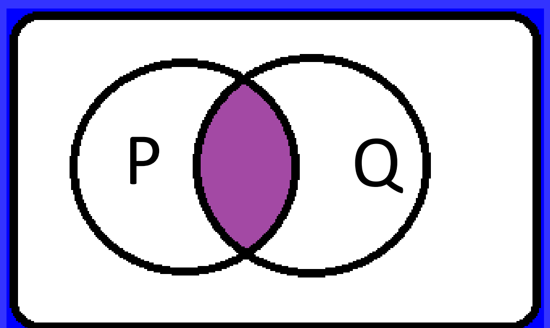
P^c = complement of P (S – P)



$P \subseteq Q$ = P is a subset of Q



$P \cup Q$ = union of P and Q



$P \cap Q$ = intersection of P and Q

Boolean algebra

What operations can we do with True and False?

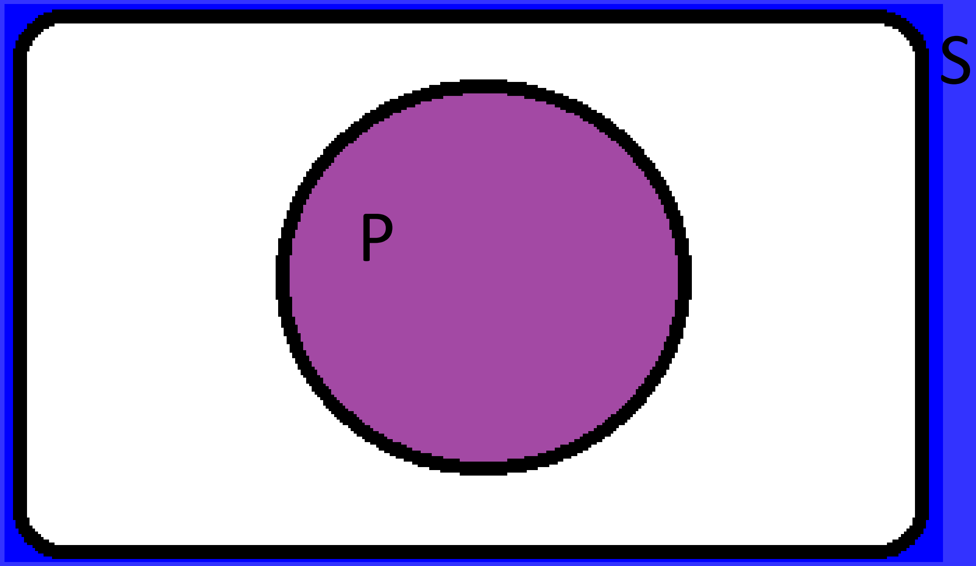
- Not ($\neg$)
- Or ($\vee$)
- And ($\wedge$)

These are the basic operations. They appear in all modern programming languages.

Boolean algebra operations

Not $(\neg)$

p	$\neg$ p
T	F
F	T

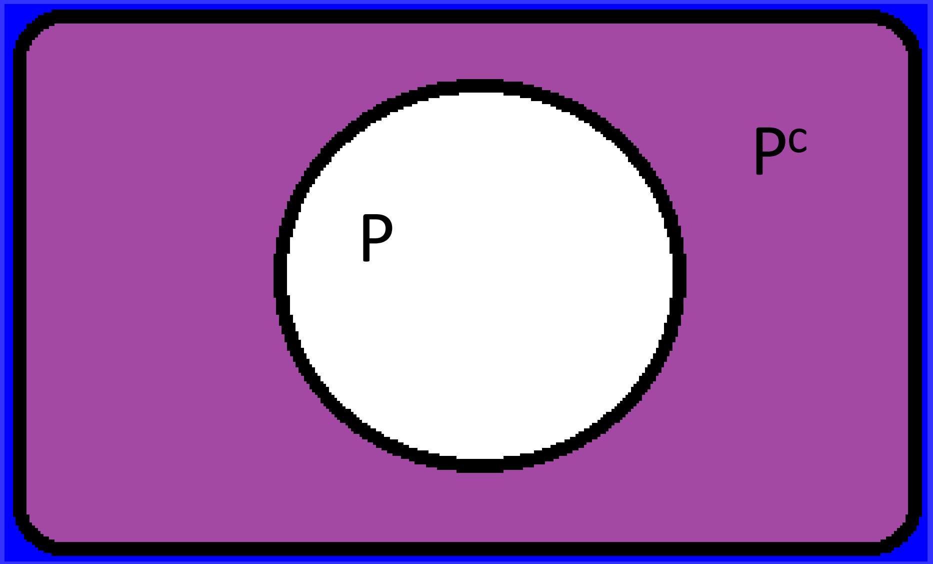


$\neg (x \in P) =$

Boolean algebra operations

Not ($\neg$)

Analogous to complement of a set



p	$\neg$ p
T	F
F	T

$\neg (x \in P) = x \in P^c$

P^c is the complement of P,

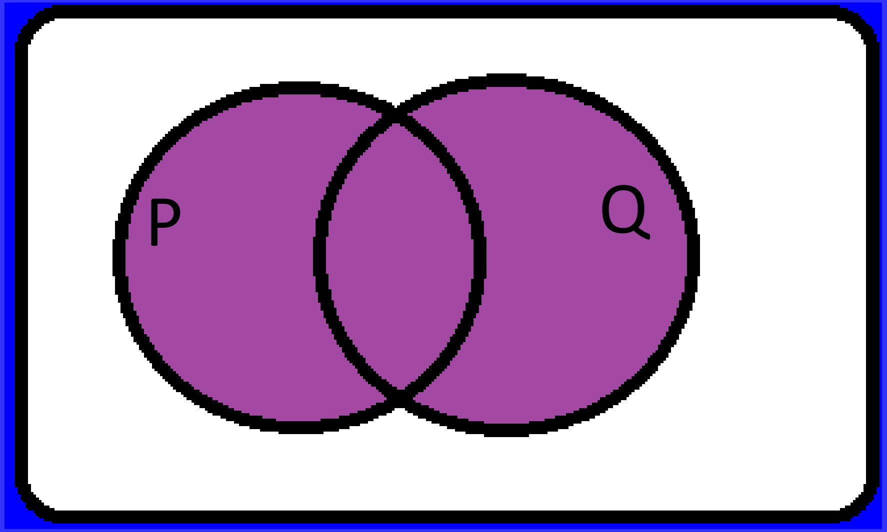
Example: P = girls in the class, P^c = boys in the class

Boolean algebra operations

Or ($\vee$), inclusive or

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Analogous to union of sets



$$(x \in P \vee x \in Q) = x \in P \cup Q$$

Example: P = boys in the class,

Q = left-handed students in the class

$P \cup Q$ = set of boys or left-handed in the class

Boolean algebra operations

Or ($\vee$)

p	q	$p \vee q$
~~T~~	~~T~~	~~T~~
~~T~~	~~F~~	~~T~~
F	T	T
~~F~~	~~F~~	~~F~~

Pam promised:

p = she would attend the school assembly

or

q = she would send a substitute.

She is not coming to the assembly,

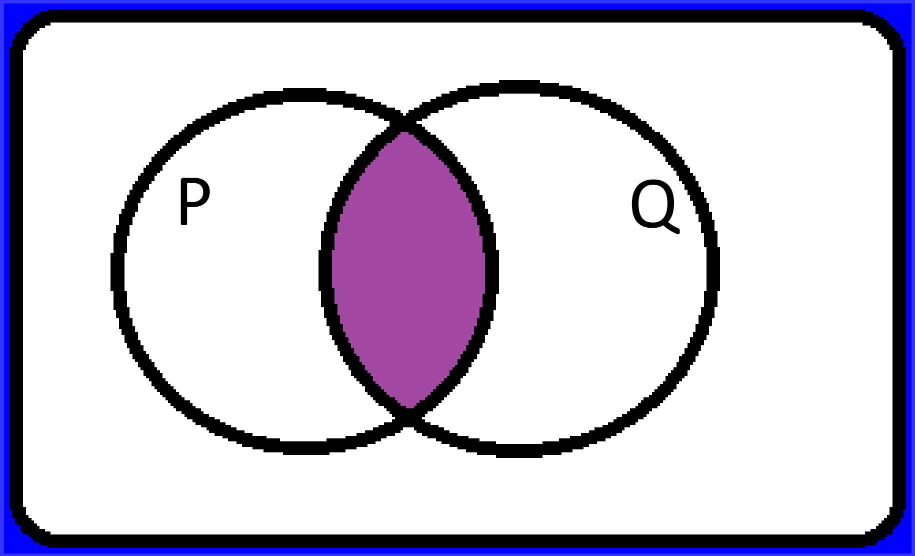
so we are expecting a substitute.

Boolean algebra operations

And ($\wedge$)

p	q	p $\wedge$ q
T	T	T
T	F	F
F	T	F
F	F	F

Analogous to intersection of sets



$$(x \in P \wedge x \in Q) = x \in P \cap Q$$

Example: P = boys in the class,

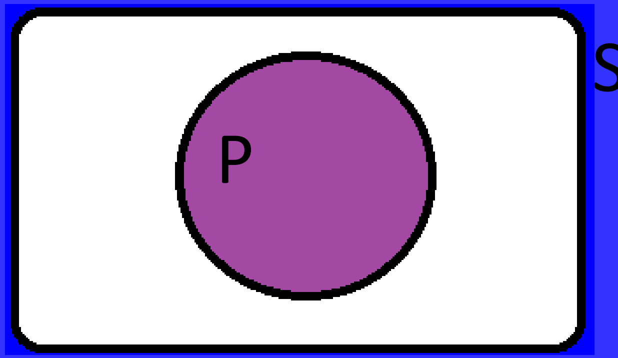
Q = left-handed students in the class

$P \cap Q$ = set of left-handed boys in the class

Indicator function

If P is a subset of S then its Boolean indicator function $\mathbf{1}_P: S \rightarrow \{\text{True}, \text{False}\}$ is:

$$\mathbf{1}_P(x) = \begin{cases} \text{True if } x \in P \\ \text{False if } x \notin P \end{cases}$$



$$\mathbf{1}_{P^c}(x) = \neg\, \mathbf{1}_P(x)$$

$$\mathbf{1}_{P \cup Q}(x) = \mathbf{1}_P(x) \vee \mathbf{1}_Q(x)$$

$$\mathbf{1}_{P \cap Q}(x) = \mathbf{1}_P(x) \wedge \mathbf{1}_Q(x)$$

$$\mathbf{1}_S(x) = \text{True}$$

$$\mathbf{1}_{\emptyset}(x) = \text{False}$$

Worksheet 2
Basic operations

1) Do you agree with the following argument?

    -I am at home or I am in the city.

    -I am at home.

    -Therefore, I am not in the city.

2) What about this one?

    -I cannot be both at home and in the city.

    -I am not at home.

    -Therefore, I am in the city.

Worksheet 2
Basic operations

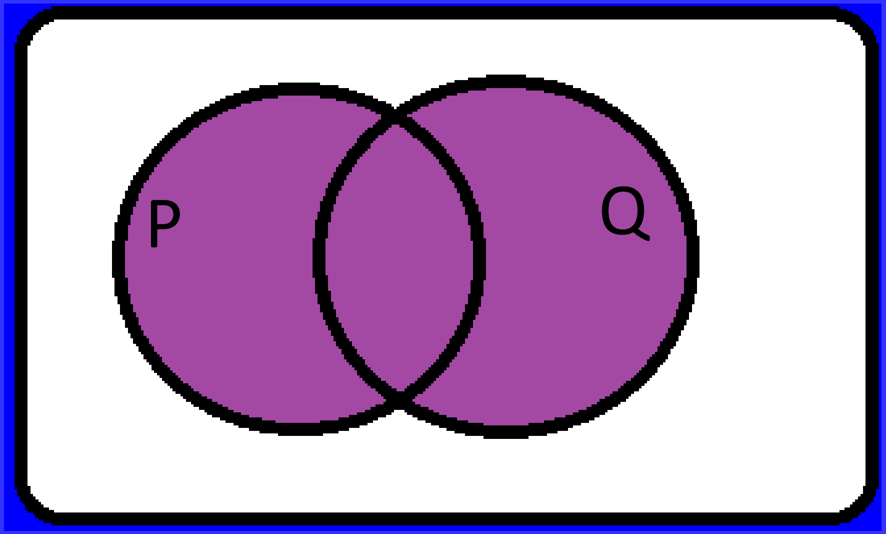
1) Do you agree with the following argument?

-I am at home or I am in the city.

-I am at home.

-Therefore, I am not in the city.

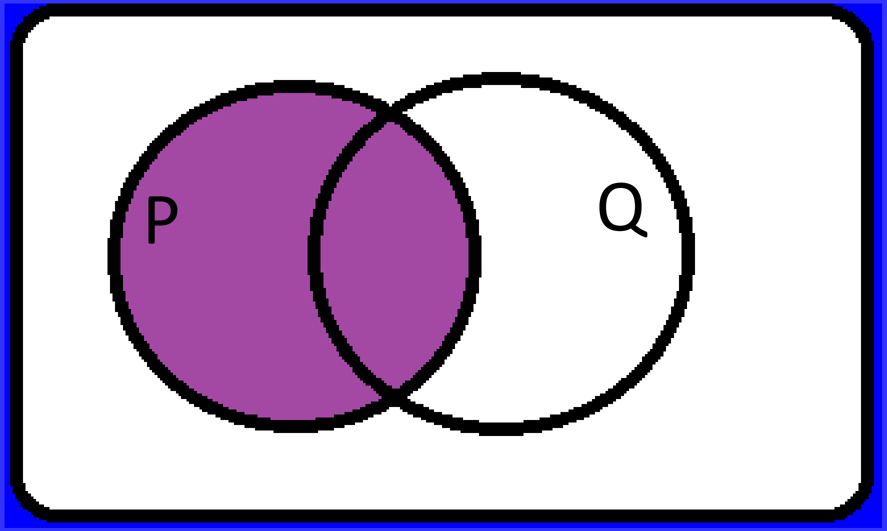
I am at home	I am in the city	I am at home or I am in the city
T	T	T
T	F	T
~~F~~	~~T~~	~~T~~
~~F~~	~~F~~	~~F~~



Worksheet 2
Basic operations

1) Do you agree with the following argument?
   - I am at home or I am in the city.
   - I am at home.
   - Therefore, I am not in the city.

I am at home	I am in the city	I am at home or I am in the city
T	T	T
T	F	T
~~F~~	~~T~~	~~T~~
~~F~~	~~F~~	~~F~~



Worksheet 2
Basic operations

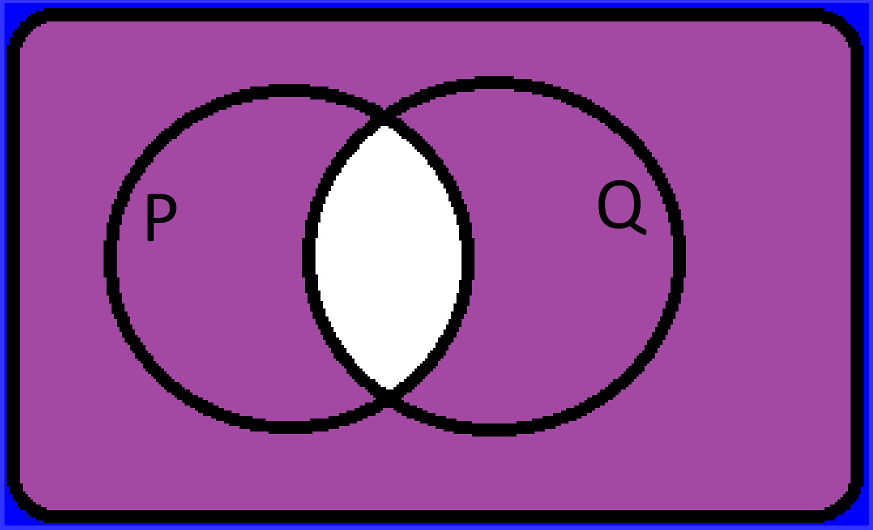
2) What about this one?

-I cannot be both at home and in the city.

-I am not at home.

-Therefore, I am in the city.

I am at home	I am in the city	I am at home and I am in the city
~~T~~	~~T~~	~~T~~
~~T~~	~~F~~	~~F~~
F	T	F
F	F	F



Worksheet 2
Basic operations

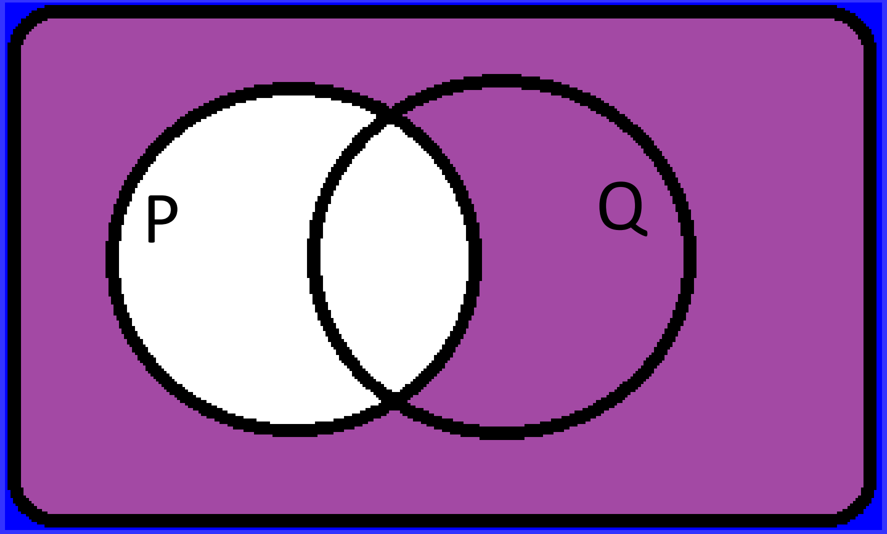
2) What about this one?

-I cannot be both at home and in the city.

-I am not at home.

-Therefore, I am in the city.

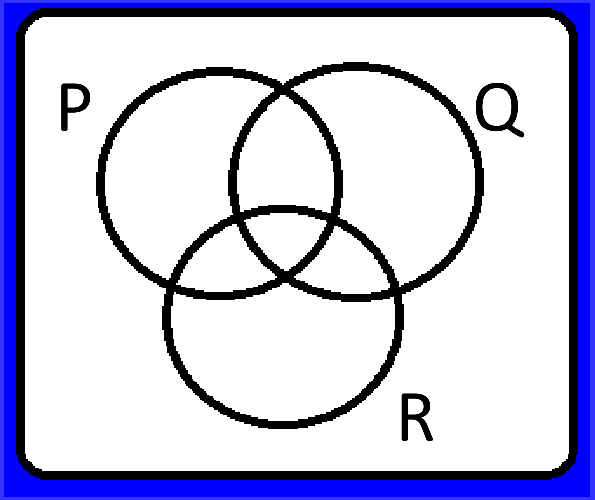
I am at home	I am in the city	I am at home and I am in the city
~~T~~	~~T~~	~~T~~
~~T~~	~~F~~	~~F~~
F	T	F
F	F	F



Worksheet 2

3) Could you draw lines pairing the expressions that are equal?

$$p \wedge (q \vee r) = (p \vee q) \wedge (p \vee r)$$



Worksheet 2

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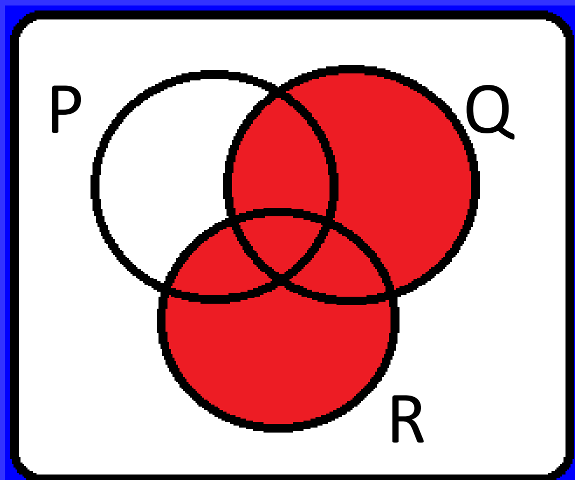
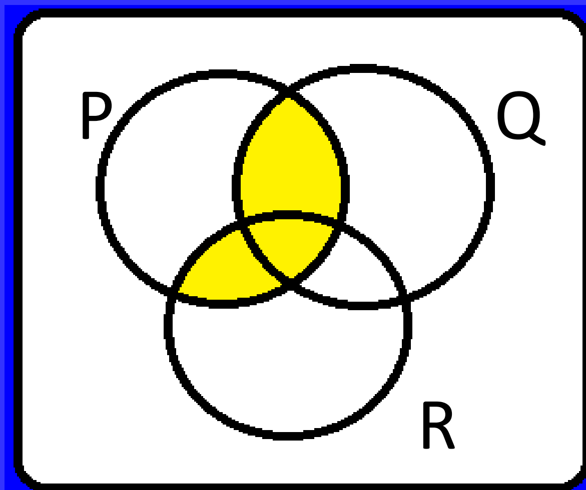
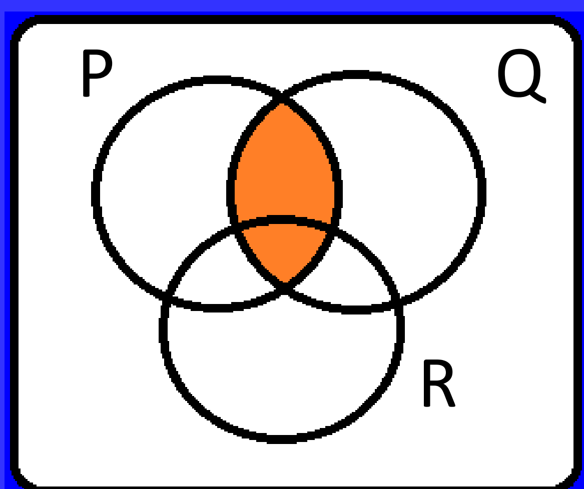
$Q \cup R$

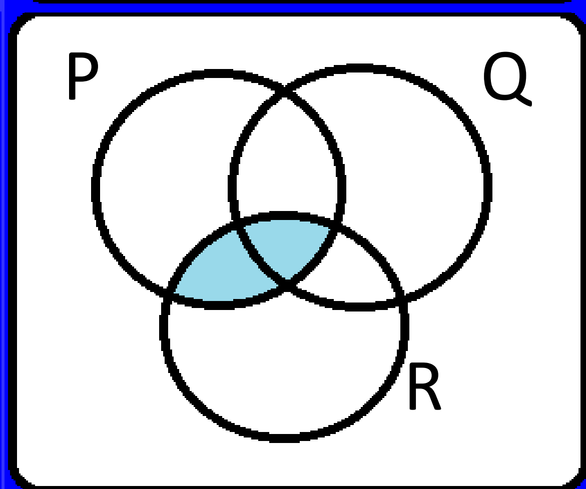




$P \cap (Q \cup R)$
$=$
$(P \cap Q) \cup (P \cap R)$

$P \cap Q$



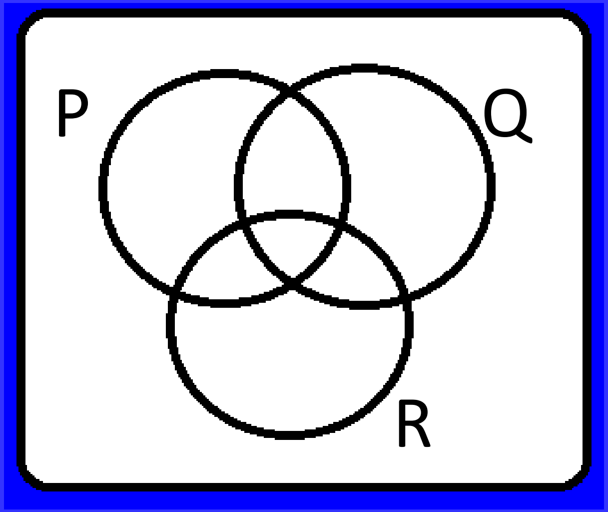


$P \cap R$

Worksheet 2

3) Could you draw lines pairing the expressions that are equal?

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$



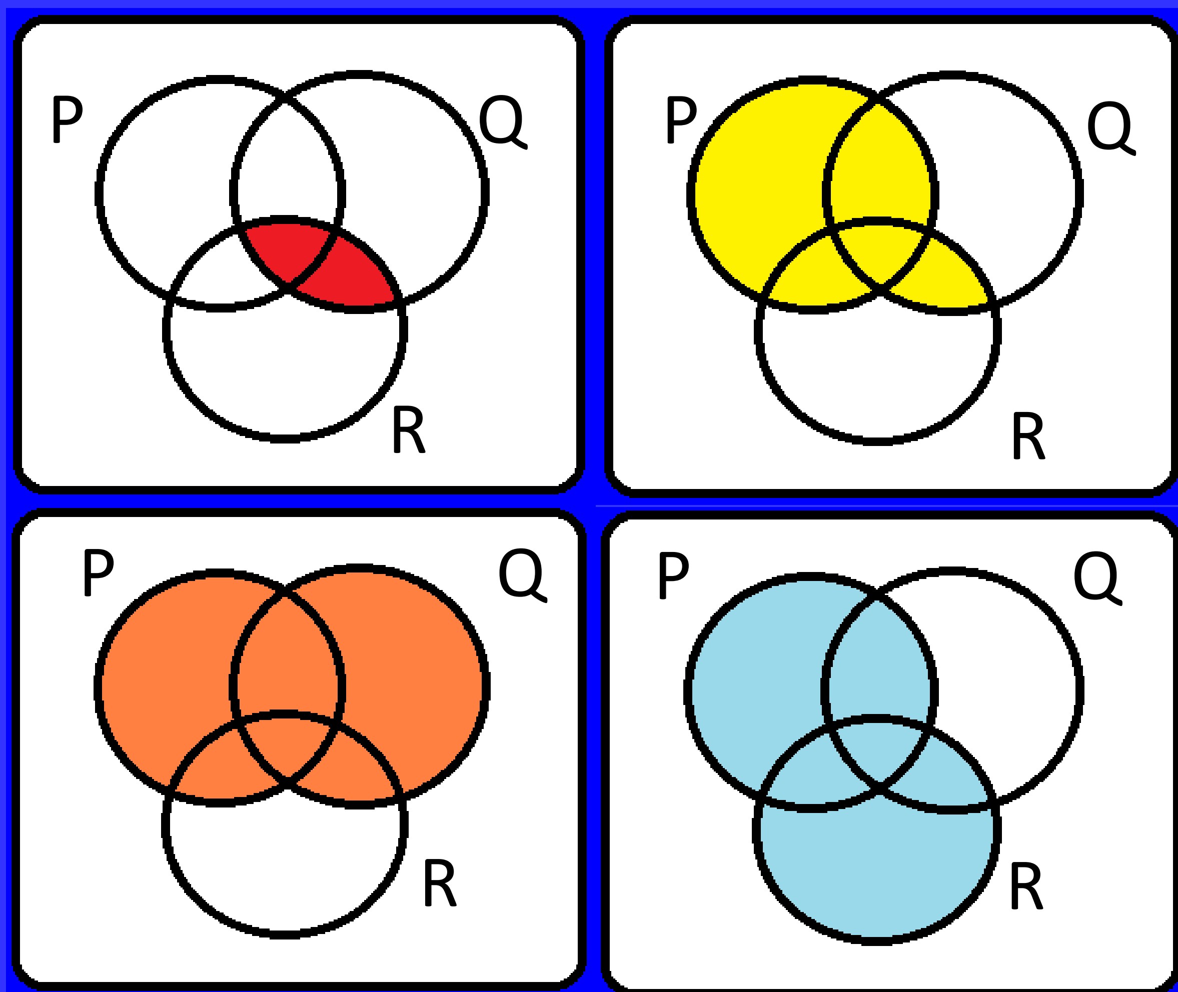
Worksheet 2

3) Could you draw lines pairing the expressions that are equal?

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

$Q \cap R$

$P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$

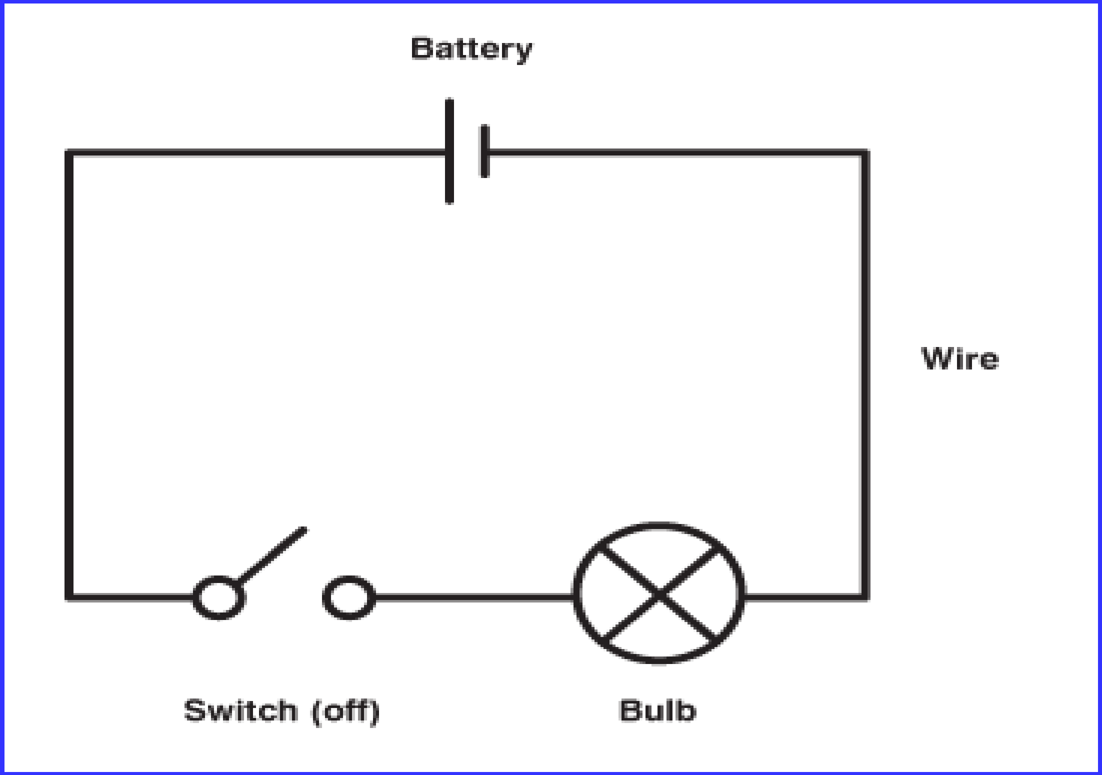


$P \cup Q$

$P \cup R$

Switching circuits

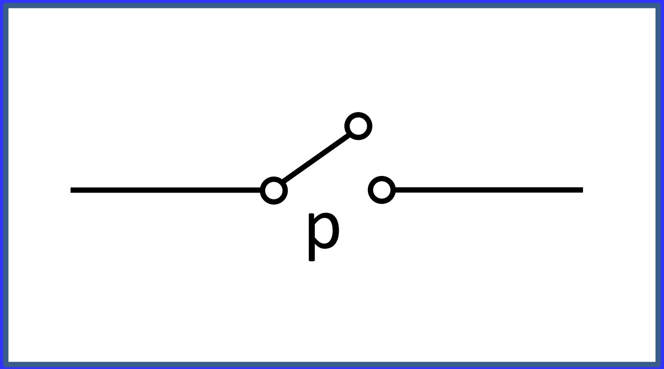
In these circuits we replace True and False by On and Off.



A switch can connect the circuit or not, that is it can be On or Off.

Switching circuits

In these circuits we replace True and False by On and Off.

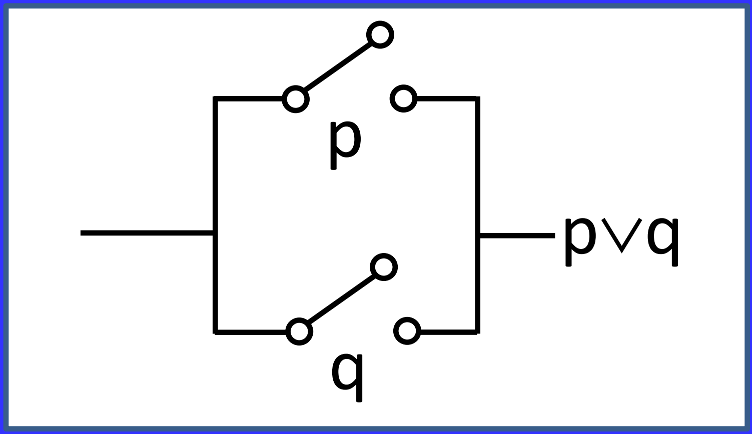


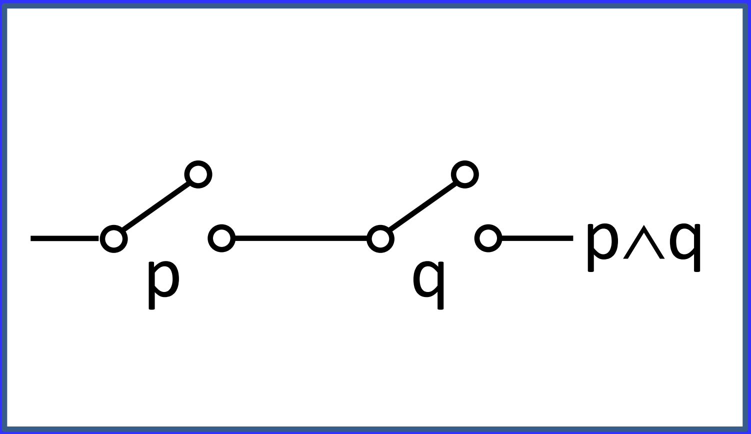
A switch p can be On or Off.

How can you connect two switches?

Switching circuits

Switching circuit operations:





Parallel

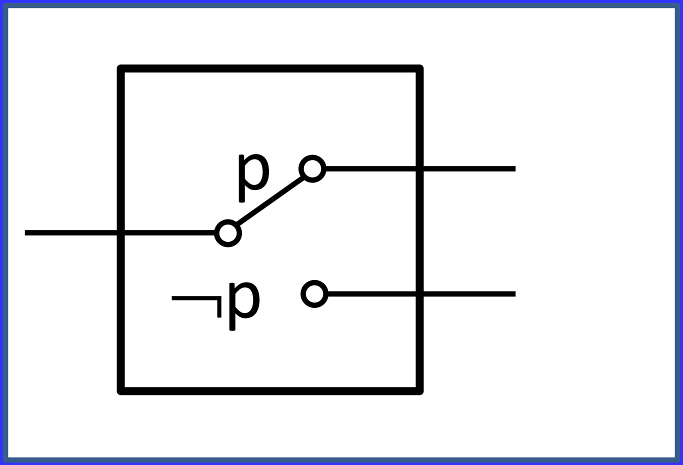
Or (∨)

Series

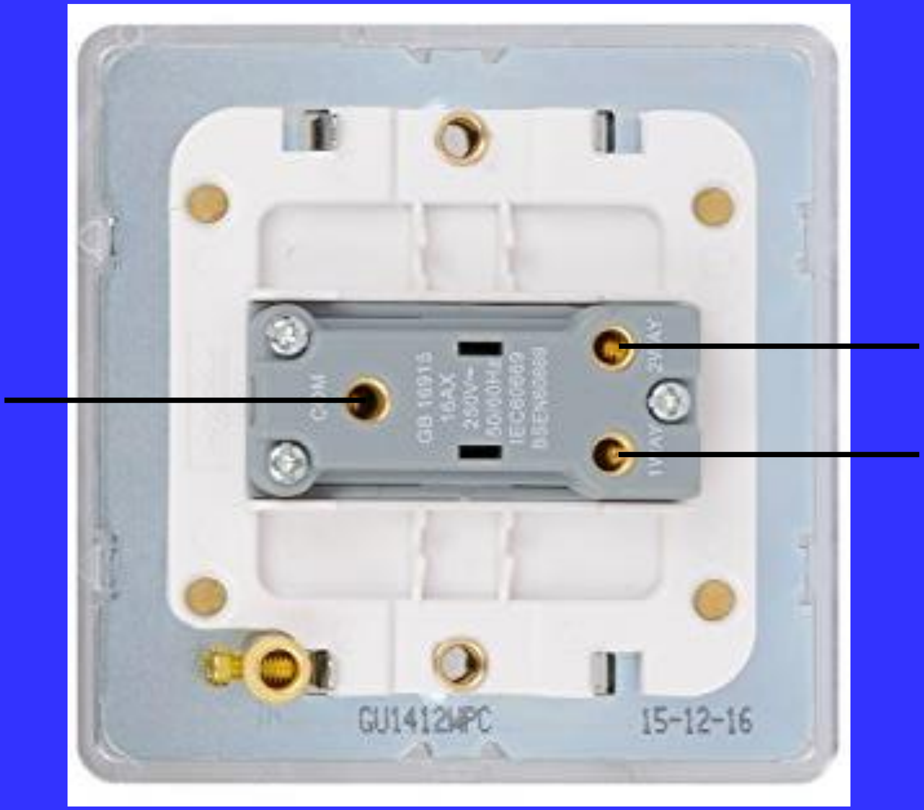
And (∧)

Switching circuits

Switching circuit operations:



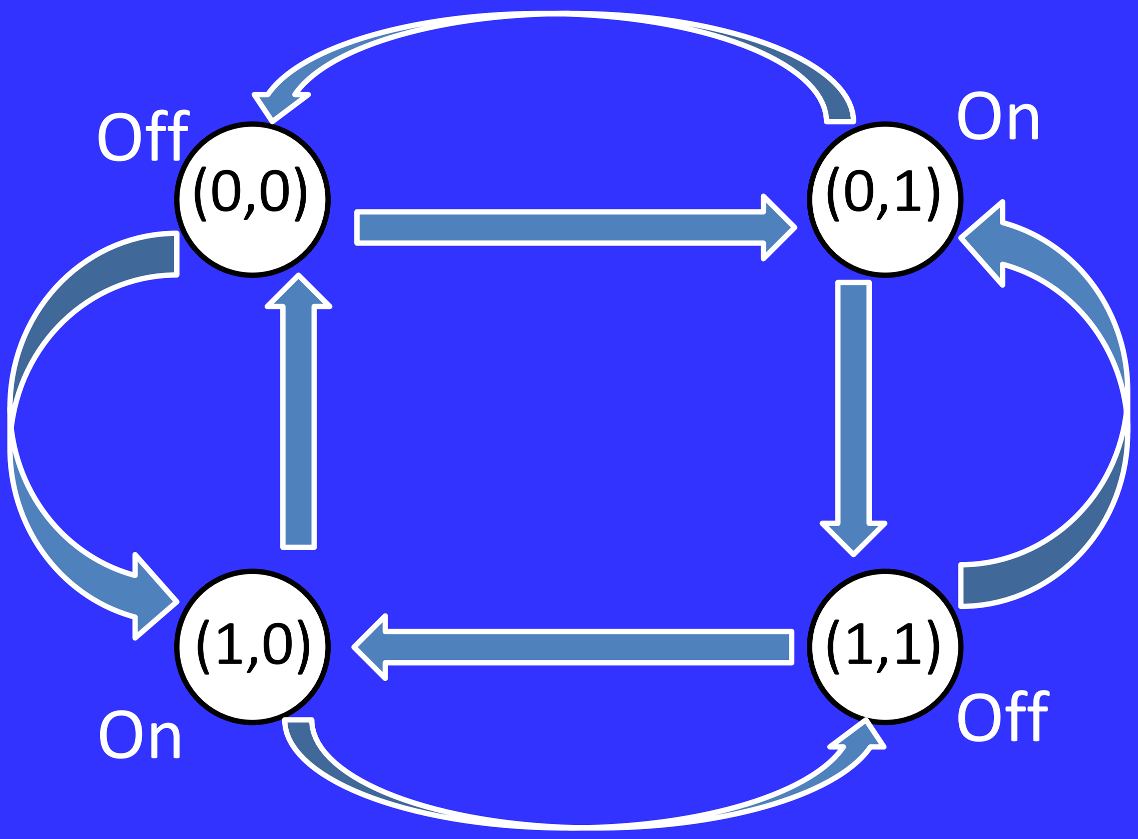
Not (¬)



A real switch

Switching circuits

What is the circuit for a light, with two switches, in a stairway?

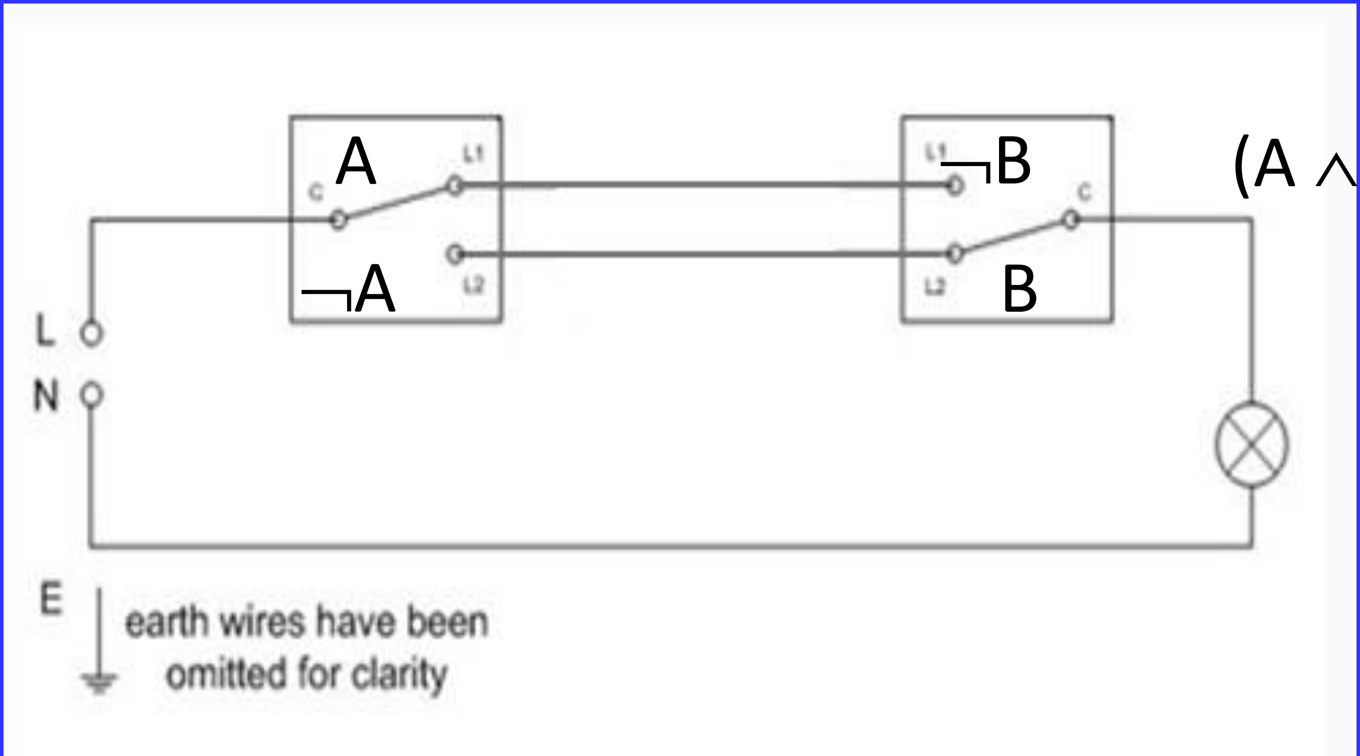


We have two switches that we represent by the state (A,B).

Therefore we want to make the light to be On when
$(A \wedge \neg B) \vee (\neg A \wedge B)$
or simply $A \neq B$.

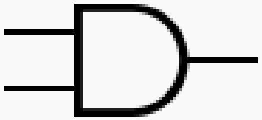
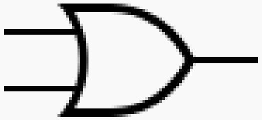
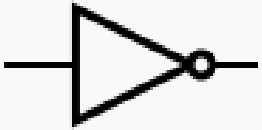
Switching circuits

What is the circuit for a light, with two switches, in a stairway?



$(A \wedge \neg B) \vee (\neg A \wedge B)$

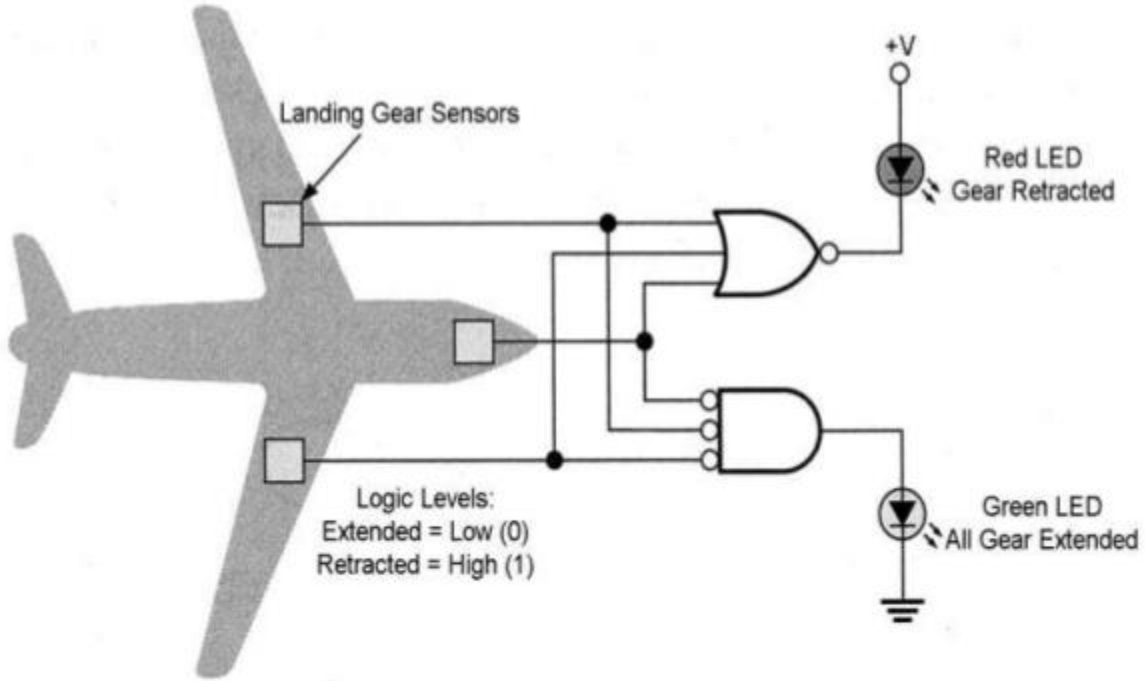
Logic gates

AND	
OR	
NOT	

Logic gates

AIRCRAFT LOGIC CIRCUIT

- **LANDING GEAR WARNING CIRCUIT**

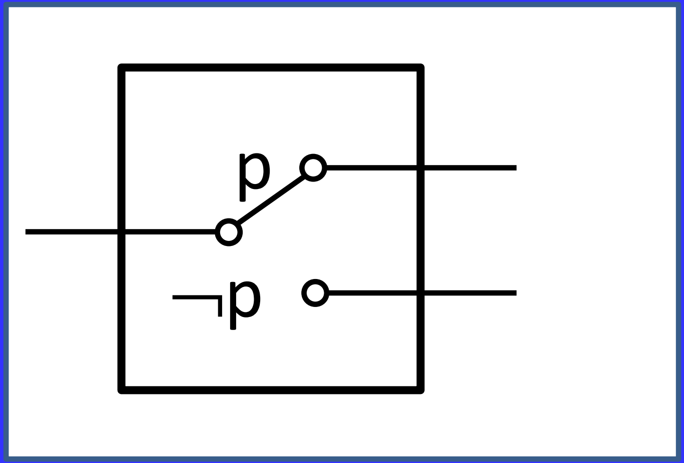


Truth tables

How to prove that $(p \vee \neg p) = T$?

p	$\neg$ p	p $\vee \neg$ p
T	F	T
F	T	T

This is called Excluded Middle.



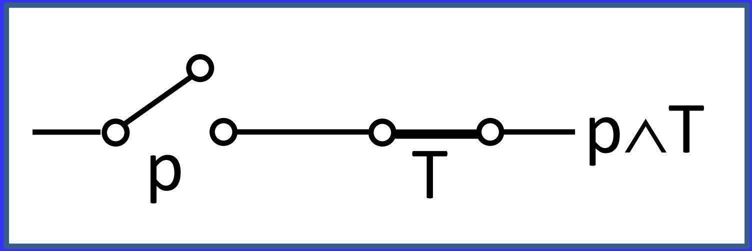
In sets, if we are talking about the subsets of a set S, the analogous to T is S:

$$P \cup P^c = S$$

Truth tables

How to prove that $(p \wedge T) = p$?

p	T	$p \wedge T$
T	T	**T**
F	T	**F**



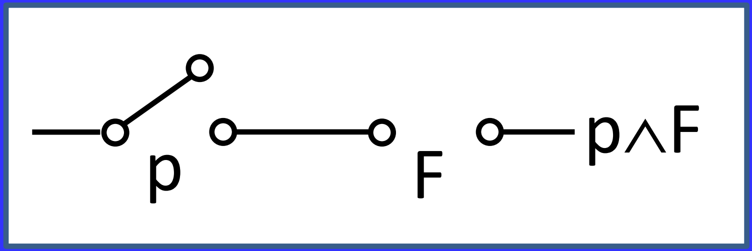
In sets:

$P \cap S = P$

Truth tables

How to prove that $(p \wedge F) = F$?

p	F	p ∧ F
T	F	F
F	F	F



In sets the analogous to F is the empty set:

$$P \cap \varnothing = \varnothing$$

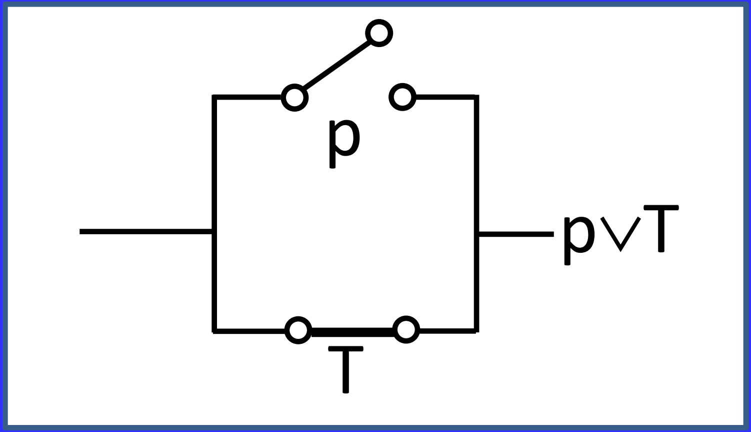
Truth tables

How to prove that $(p \vee T) = T$?

p	T	$p \vee T$
T	T	T
F	T	T

In sets:

$$P \cup S = S$$



Truth tables

How to prove that $\neg(p \vee q) = \neg p \wedge \neg q$?

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	**F**
T	F	T	**F**
F	T	T	**F**
F	F	F	**T**

$\neg p$	$\neg q$	$\neg p \wedge \neg q$
F	F	**F**
F	T	**F**
T	F	**F**
T	T	**T**

This is one De Morgan's law.

Worksheet 3
Truth tables

1) Using a truth table prove that $(p \wedge \neg p) = \text{False}$. This is called the "no contradiction" law.

2) Using a truth table prove that $(p \vee \text{False}) = p$.

3) Using a truth table prove that $\neg (p \wedge q) = \neg p \vee \neg q$. This is a De Morgan's law.

Worksheet 3
Truth tables

Remember:

How to prove that $\neg(p \vee q) = \neg p \wedge \neg q$?

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	**F**
T	F	T	**F**
F	T	T	**F**
F	F	F	**T**

$\neg p$	$\neg q$	$\neg p \wedge \neg q$
F	F	**F**
F	T	**F**
T	F	**F**
T	T	**T**

Worksheet 3
Truth tables

1) Using a truth table prove that $(p \wedge \neg p)$ = False. This is called the "no contradiction" law.

p	$\neg$ p	p $\wedge \neg$ p
T	F	F
F	T	F

Worksheet 3
Truth tables

2) Using a truth table prove that $(p \vee \text{False}) = p$.

p	**F**	**$p \vee F$**
T	F	**T**
F	F	**F**

Worksheet 3
Truth tables

1) Using a truth table prove that $\neg(p \wedge q) = \neg p \vee \neg q$. This is a De Morgan's law.

p	**q**	**$p \wedge q$**	**$\neg(p \wedge q)$**
T	T	T	**F**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

$\neg p$	**$\neg q$**	**$\neg p \vee \neg q$**
F	F	**F**
F	T	**T**
T	F	**T**
T	T	**T**

Boolean algebra

What other operations can we do with True and False?

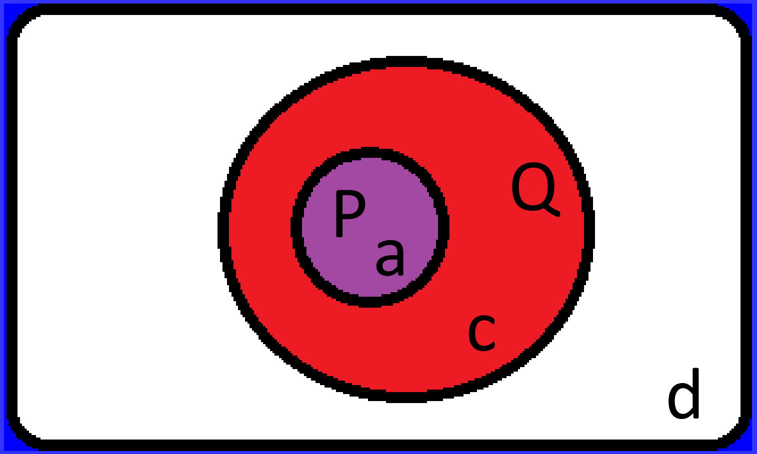
- Implication ($\Rightarrow$)

Boolean algebra operations

Implication ($\Rightarrow$)

p	q	p $\Rightarrow$ q	Case
T	T	T	a
T	F		
F	T	T	c
F	F	T	d

Analogous to subset relationship



$$(x \in P \Rightarrow x \in Q) = (P \subseteq Q)$$

If there are delays in the Circle line then Jenny is late at work.

(There are delays in the Circle line $\Rightarrow$ Jenny is late at work) = ($P \subseteq Q$)

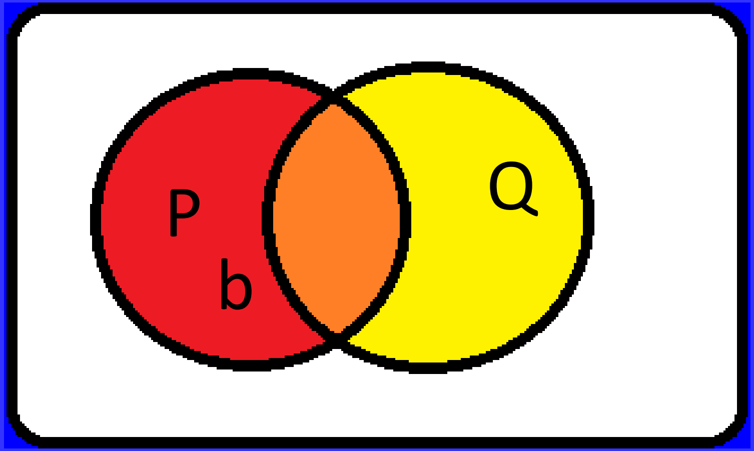
Example: P = {mornings when there are delays in the Circle line },
Q = {mornings when Jenny is late at work}

Boolean algebra operations

Implication ($\Rightarrow$)

p	q	$p \Rightarrow q$	Case
T	T	T	a
T	F	F	b
F	T	T	c
F	F	T	d

Analogous to subset relationship



$$(x \in P \Rightarrow x \in Q) = (P \subseteq Q)$$

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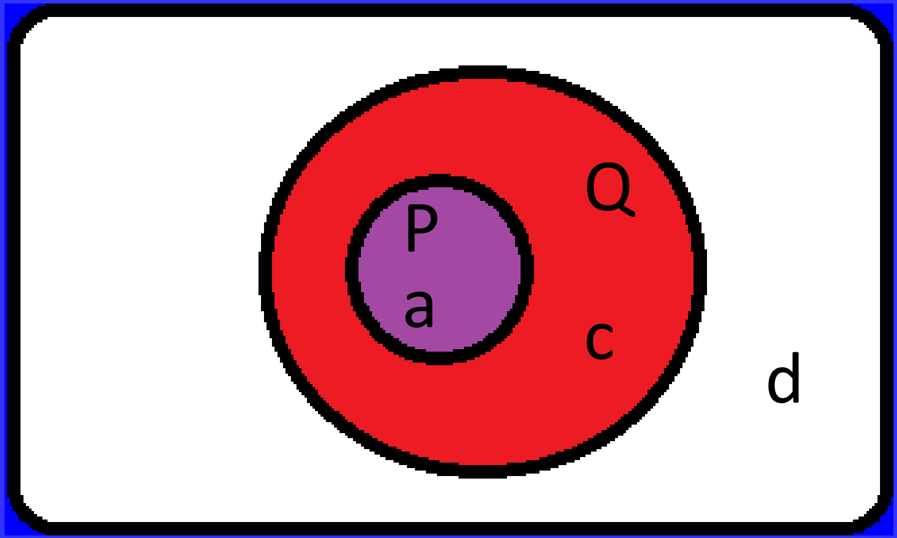
Q = {mornings when Jenny is late at work}

Boolean algebra operations

Implication ($\Rightarrow$)

p	q	$p \Rightarrow q$	Case
T	T	T	a
T	F	F	b
F	T	T	c
F	F	T	d

Analogous to subset relationship



$$(x \in P \Rightarrow x \in Q) = (P \subseteq Q)$$

Note that $P \subseteq Q$ is not a set so $\subseteq$ is not a set operation,

but $p \Rightarrow q$ is True or False so $\Rightarrow$ is a Logic operation.

Boolean algebra operations

Implication ($\Rightarrow$)

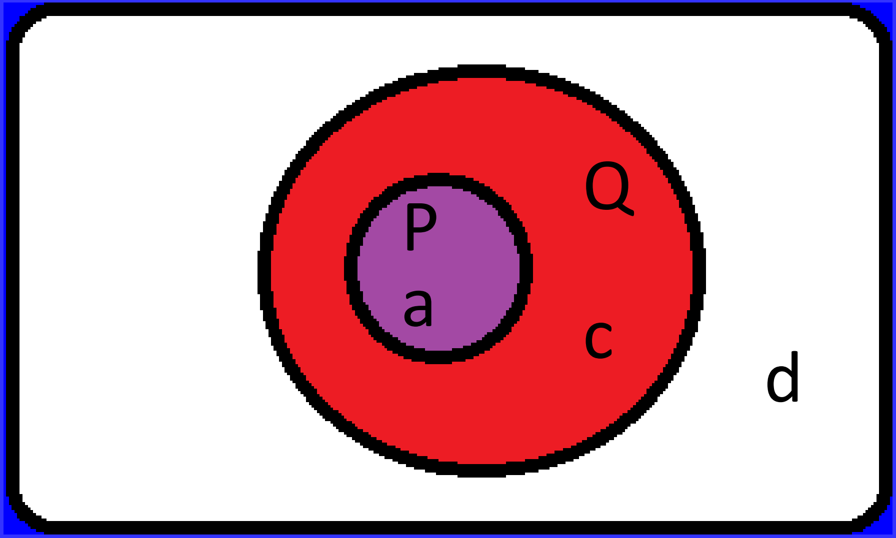
p	q	$p \Rightarrow q$	Case
T	T	T	a
T	F	F	b
F	T	T	c
F	F	T	d

$$x \in P \Rightarrow x \in Q$$

$$(p \Rightarrow q) = (p = (p \wedge q))$$

$$(p \Rightarrow q) = (q = (p \vee q))$$

Analogous to subset relationship



$$P \subseteq Q$$

$$(P \subseteq Q) = (P = P \cap Q)$$

$$(P \subseteq Q) = (Q = P \cup Q)$$

Truth tables

How to prove that $(p \Rightarrow q) = (p = (p \wedge q))$?

p	**q**	**$p \Rightarrow q$**
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

$p \wedge q$	**$p = (p \wedge q)$**
T	**T**
F	**F**
F	**T**
F	**T**

Truth tables

If there are delays in the Circle line then Jenny is late at work. Today she has arrived on time therefore there are no delays in the Circle line.

How to prove that $(p \Rightarrow q) = (\neg q \Rightarrow \neg p)$?

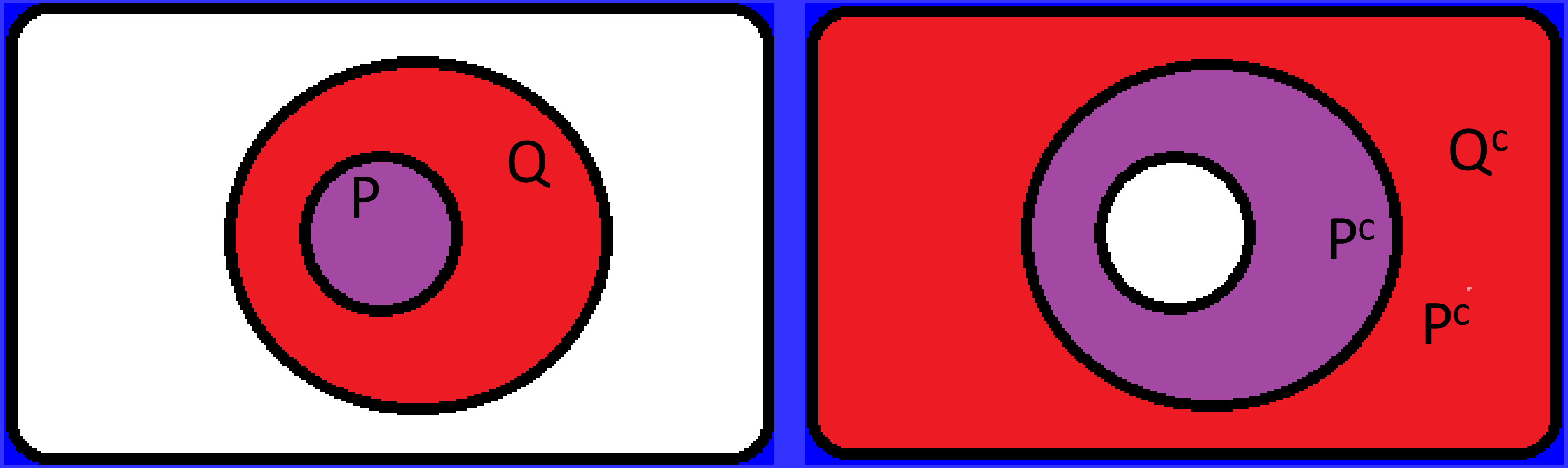
p	q	$p \Rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
F	F	**T**
T	F	**F**
F	T	**T**
T	T	**T**

p is sufficient for q, q is necessary for p

Analogy with sets

How to prove that $(p \Rightarrow q) = (\neg q \Rightarrow \neg p)$?



$$(P \subseteq Q) = (Q^c \subseteq P^c)$$

Worksheet 4
& Break!

Worksheet 4
Implication

1) Peter's mom said: "All champions are good at maths." Peter says: "I am good at maths. Therefore I am a champion!" Is his reasoning correct?

2) There are four different cards on a table; each one has a letter on one side and a natural number on the other one. The visible sides have the symbols A, B, 4, and 5. What is the minimum number of cards we must turn over to find out whether the following statement is true: "If an even number is written on one side of a card then a vowel is written on the other side"? Which cards we need to turn over?

A	B	4	5

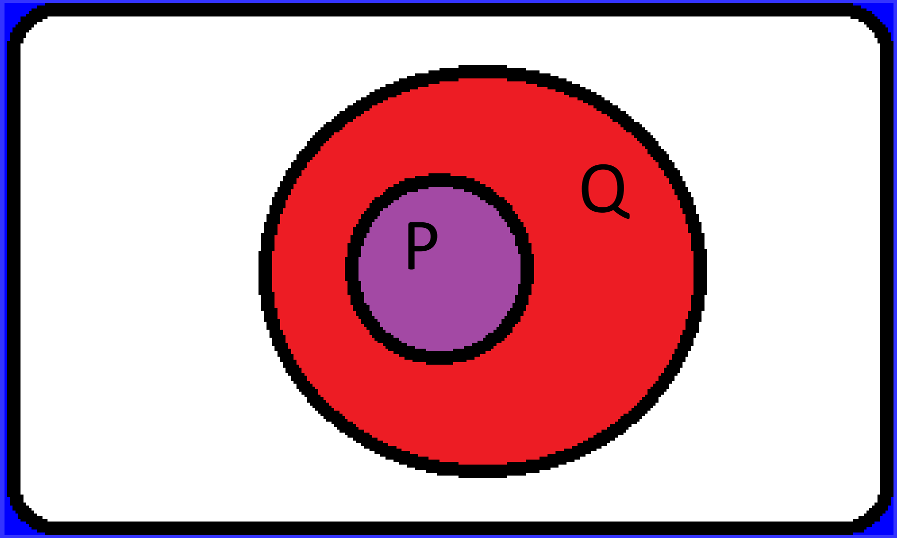
Worksheet 4
Implication

Remember:

Implication ($\Rightarrow$)

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

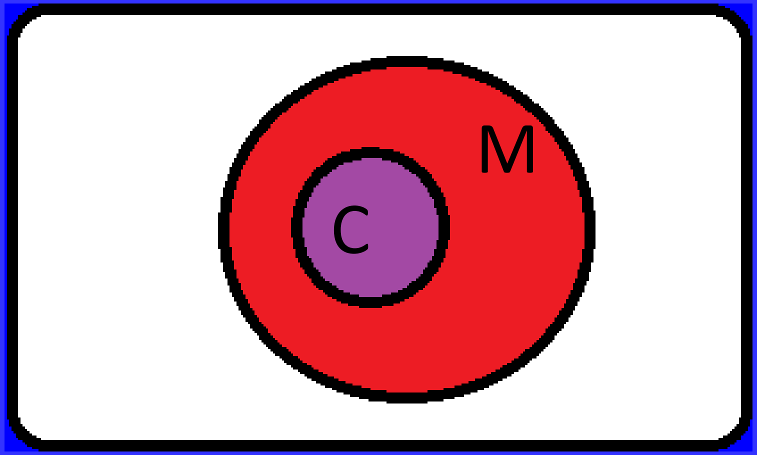
Analogous to subset relationship



$$(x \in P \Rightarrow x \in Q) = (P \subseteq Q)$$

Worksheet 4
Implication

1) Peter's mom said: "All champions are good at maths." Peter says: "I am good at maths. Therefore I am a champion!" Is his reasoning correct?



Which set is C and which is M?

Worksheet 4
Implication

2) There are four different cards on a table; each one has a letter on one side and a natural number on the other one. The visible sides have the symbols A, B, 4, and 5. What is the minimum number of cards we must turn over to find out whether the following statement is true: "If an even number is written on one side of a card then a vowel is written on the other side"? Which cards we need to turn over?

A	B	4	5

Even	Vowel	Even $\Rightarrow$ Vowel
T	T	T
T	F	F
F	T	T
F	F	T

Worksheet 4
Implication

3) Using a truth table prove that $(p \Rightarrow q) \neq (q \Rightarrow p)$. This proves that *"If there are delays in the Circle line then Jenny is late at work. Today she has arrived late therefore there are delays in the Circle line."* is incorrect.

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	**T**	**T**
T	F	**F**	**T**
F	T	**T**	**F**
F	F	**T**	**T**

Worksheet 4
Implication

4) Using a truth table prove that $(p \Rightarrow q) = (\neg p \vee q)$.

p	q	$p \Rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

$\neg p$	$\neg p \vee q$
F	**T**
F	**F**
T	**T**
T	**T**

Worksheet 4
Implication

5) Using a truth table prove that $(p \Rightarrow q) = (q = (p \vee q))$.

p	q	$p \Rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

$p \vee q$	$q = (p \vee q)$
T	**T**
T	**F**
T	**T**
F	**T**

Boolean algebra

What other operations can we do with True and False?

- Implication ($\Longrightarrow$)

  Also known as conditional:

  if p then q

- Equality ($=$)

  Also known as biconditional ($\Longleftrightarrow$)

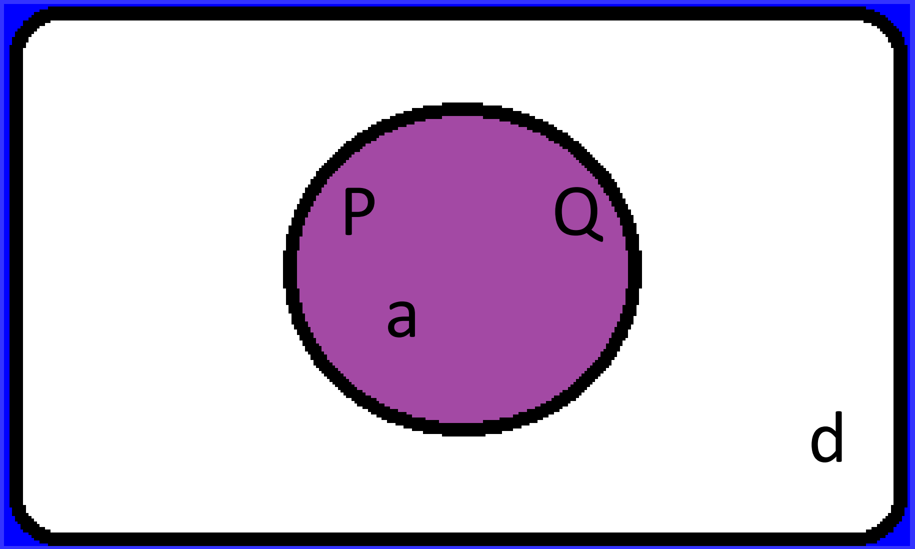
  p if and only if q

Boolean algebra operations

Equality (=)

p	q	p = q	Case
T	T	T	a
T	F	F	
F	T	F	
F	F	T	d

Analogous to equality of sets



$$(x \in P = x \in Q) = (P = Q)$$

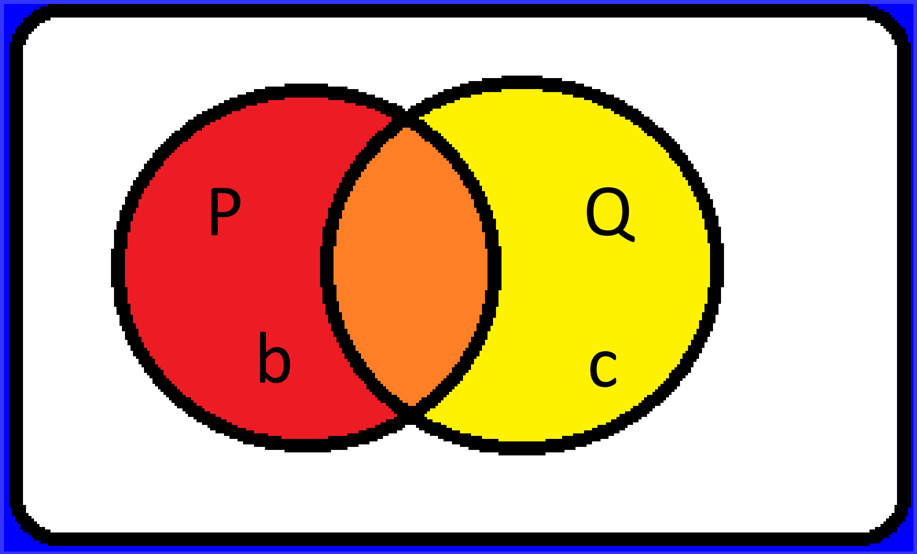
Note that P = Q is not a set so = is not a set operation, but p = q is True or False so = is a Logic operation.

Boolean algebra operations

Equality (=)

p	q	p = q	Case
T	T	T	a
T	F	F	b
F	T	F	c
F	F	T	d

Analogous to equality of sets



$$(x \in P = x \in Q) = (P = Q)$$

Note that P = Q is not a set so = is not a set operation, but p = q is True or False so = is a Logic operation.

Equality is associative

Sum of numbers is associative: $(a + b) + c = a + (b + c)$

Equality is associative: $((p = q) = r) = (p = (q = r))$

p	q	r	p = q	(p = q) = r
T	T	T	T	**T**
T	T	F	T	**F**
T	F	T	F	**F**
T	F	F	F	**T**
F	T	T	F	**F**
F	T	F	F	**T**
F	F	T	T	**T**
F	F	F	T	**F**

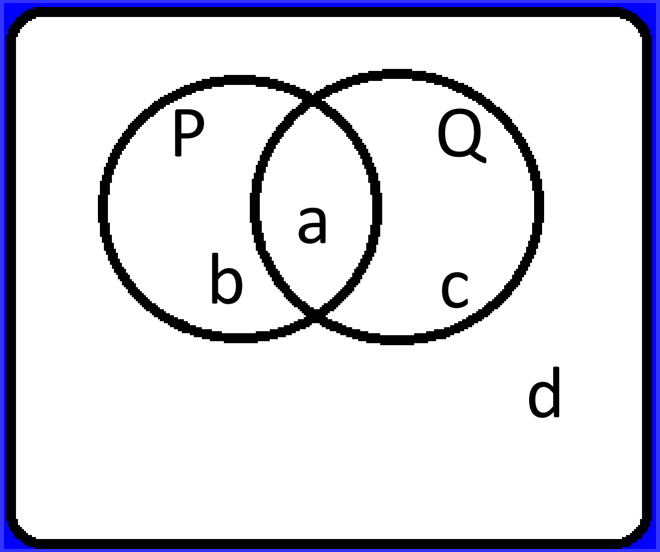
q = r	p = (q = r)
T	**T**
F	**F**
F	**F**
T	**T**
T	**F**
F	**T**
F	**T**
T	**F**

Boolean algebra operations

Another analogy,
a new set operation:

Equality (=)

p	q	p = q	Case
T	T	T	a
T	F	F	b
F	T	F	c
F	F	T	d



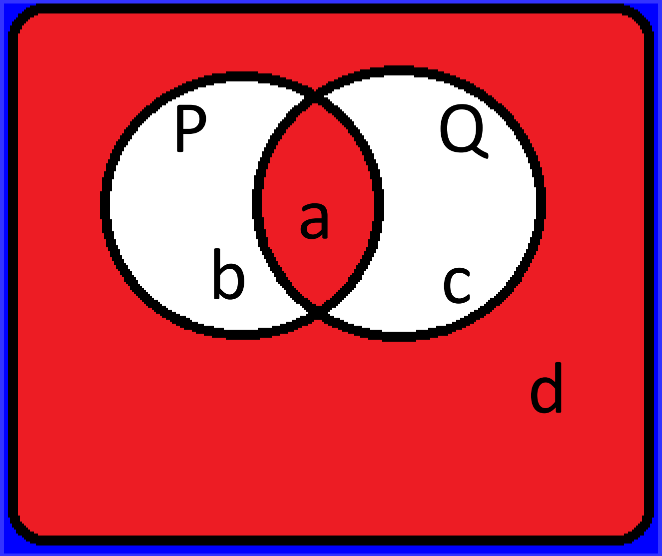
For which regions is true that $x \in P = x \in Q$?

Boolean algebra operations

Equality (=)

p	q	p = q	Case
T	T	T	a
T	F	F	b
F	T	F	c
F	F	T	d

Another analogy,
a new set operation:
both or none

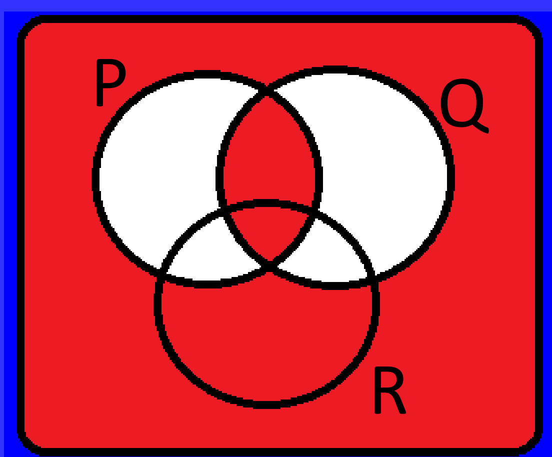


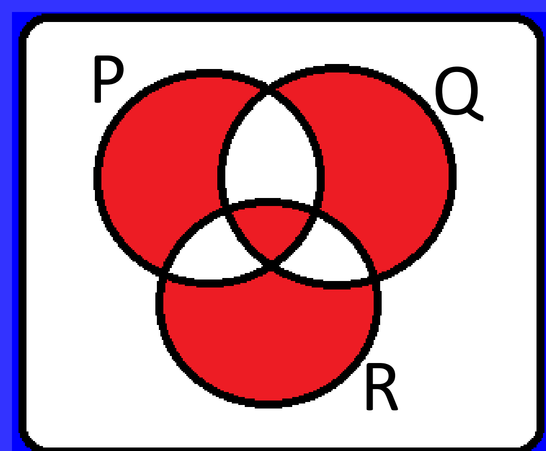
For which regions is true that $x \in P = x \in Q$?

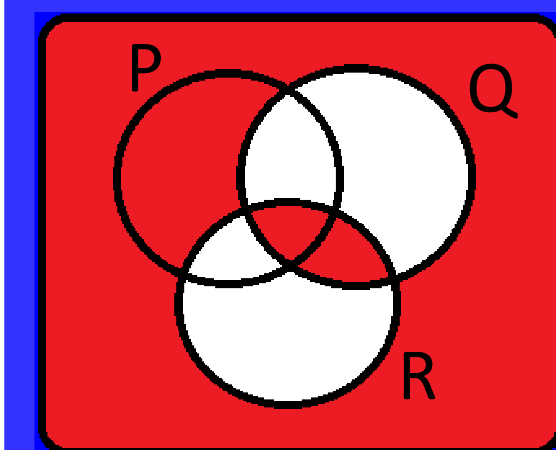
Equality is associative

Equality is associative: $((p = q) = r) = (p = (q = r))$

Equality as a set operation: both or none







Arithmetic and Algebra

I bought some goods paying £910 in cash, using exactly 23 notes of £20 and £50. How many £50 notes did I use?

Arithmetic way (using the Regula Falsi method):

If the 23 notes were of £20 the total would have been 23*£20 = £460. That is off by £910 - £460 = £450.

For each £20 note that we replace by a £50 one the total increases by £30.

∴ I used £450/£30 = 15 £50 notes.

Algebraic way:

x = number of £20 notes

y = number of £50 notes

$$x + y = 23 \quad (1)$$

$$20x + 50y = 910 \quad (2)$$

$$20x + 20y = 460 \quad (20*(1))$$

$$30y = 450 \quad (2) - 20*(1)$$

$$y = 15 \quad \text{(divide by 30)}$$

∴ I used 15 £50 notes.

Logic and Algebra

There is an island with two types of people: knights, who always tell the truth, and liars, who always lie. One inhabitant, A, tells you "I am the same type as B". What can you conclude from that?

Case analysis (traditional) way:

If A is a knight then "I am the same type as B" is true and B is a knight.

If A is a liar then "I am the same type as B" is false and B is a knight.

Given that A is a knight or a liar we conclude that B is a knight.

Algebraic (calculational) way:

-Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

-If A lives on the island and says S that means $\mathcal{A}$ = S.

Knights and liars

Algebraic key step:

-Suppose A means "A is a knight".

-If A lives on the island and says S, what do we know?

If A is a knight then S is true

$A \Rightarrow S$

If A is a liar then S is false

$\neg A \Rightarrow \neg S$

$\neg\neg S \Rightarrow \neg\neg A$

$S \Rightarrow A$

$A \Leftrightarrow S$

$A = S$

$(p \Rightarrow q) = (\neg q \Rightarrow \neg p)$

$\neg\neg p = p$

Logic and Algebra

There is an island with two types of people: knights, who always tell the truth, and liars, who always lie. One inhabitant, A, tells you "I am the same type as B". What can you conclude from that?

Case analysis (traditional) way:

If A is a knight then "I am the same type as B" is true and B is a knight.

If A is a liar then "I am the same type as B" is false and B is a knight.

Given that A is a knight or a liar we conclude that B is a knight.

Algebraic (calculational) way:

-Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

-If A lives on the island and says S that means $\mathcal{A}$ = S.

-A said ($\mathcal{A} = \mathcal{B}$), therefore

$$\mathcal{A} = (\mathcal{A} = \mathcal{B})$$

$$(\mathcal{A} = \mathcal{A}) = \mathcal{B} \quad \text{(associativity of =)}$$

$$\text{True} = \mathcal{B} \quad (\mathcal{A} = \mathcal{A} \text{ is True})$$

$\therefore$ B is a knight.

Knights and liars

There is an island with two types of people: knights, who always tell the truth, and liars, who always lie. One person, A, tells you “I am the same type as B”. What can you conclude from that?

Truth table way: $\mathcal{A}$ means “A is a knight” and $\mathcal{B}$ means “B is a knight”.

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} = \mathcal{B}$	Could A say ($\mathcal{A} = \mathcal{B}$)?
T	T	T	Yes
T	F	F	No
F	T	F	Yes
F	F	T	No

∴ B is a knight

∴ We can’t conclude anything about A.

Knights and liars

One inhabitant of the island, A, says:

If B is a knight, then I am a liar. A says $(\mathcal{B} \Rightarrow \neg\mathcal{A})$

What are A and B?

Algebraic way:

-Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

-If A lives on the island and says S that means $\mathcal{A}$ = S.

Given the implication, in order to use the associativity of = we recall: $(p \Rightarrow q) = (p = (p \wedge q))$

$(p \Rightarrow q) = (q = (p \vee q))$

$\mathcal{A} = (\mathcal{B} \Rightarrow \neg\mathcal{A})$

$\mathcal{A} = (\neg\mathcal{A} = (\mathcal{B} \vee \neg\mathcal{A}))$

$(\mathcal{A} = \neg\mathcal{A}) = (\mathcal{B} \vee \neg\mathcal{A})$ (assoc. of =)

False $= (\mathcal{B} \vee \neg\mathcal{A})$ ($\mathcal{A}=\neg\mathcal{A}$ is False)

True $= \neg(\mathcal{B} \vee \neg\mathcal{A})$ (negate both sides)

True $= (\neg\mathcal{B} \wedge \mathcal{A})$ (De Morgan)

$\therefore$ $\mathcal{B}$ = False and $\mathcal{A}$ = True

$\therefore$ A is a knight and B is a liar.

Worksheet 5
Knights and liars

Remember: Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight". If A lives on the island and says S then we only know that $\mathcal{A}$ = S.

1) While visiting the Knights and Liars Island, you meet a boy who tells you that he is a liar. Does he live on the island or is he a tourist?
2) The world is divided into two species, vampires and werewolves. Vampires always tell the truth when talking about a vampire, but always lie when talking about a werewolf. Werewolves always tell the truth when talking about a werewolf, but always lie when talking about a vampire. A says "B is a werewolf." Explain why A must be a werewolf.

Worksheet 5
Knights and liars

Remember: There is an island with two types of people: knights, who always tell the truth, and liars, who always lie. One inhabitant, A, tells you "I am the same type as B". What can you conclude from that?

Case analysis (traditional) way:

If A is a knight then "I am the same type as B" is true and B is a knight.

If A is a liar then "I am the same type as B" is false and B is a knight.

Given that A is a knight or a liar we conclude that B is a knight.

Algebraic (calculational) way:

-Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

-If A lives on the island and says S that means $\mathcal{A}$ = S.

-A said $(\mathcal{A} = \mathcal{B})$, therefore

$\mathcal{A} = (\mathcal{A} = \mathcal{B})$

$(\mathcal{A} = \mathcal{A}) = \mathcal{B}$ (associativity of =)

$\text{True} = \mathcal{B}$ ($\mathcal{A} = \mathcal{A}$ is True)

$\therefore$ B is a knight.

Worksheet 5 - Knights and liars

1) While visiting the Knights and Liars island, you meet a boy who tells you that he is a liar. Does he live on the island or is he a tourist?

Case analysis way:

If the boy is a knight then he would be lying when he says that he is a liar. That is a contradiction.

If he is a liar then he would be telling the truth when he says that he is a liar. That is a contradiction.

Therefore he can't live on the island, that is he is a tourist.

Algebraic way :

- Suppose $\mathcal{B}$ means "B is a knight".

-If B lives on the island and says S that means $\mathcal{B}$ = S.

-B said ($\mathcal{B} =$ False). If B lives on the island we would have

$\mathcal{B} = (\mathcal{B} =$ False$)$

$(\mathcal{B} = \mathcal{B}) =$ False (associativity of =)

True $=$ False ($\mathcal{B}$ = $\mathcal{B}$ is True)

A contradiction, $\therefore$ B is a tourist.

Worksheet 5 - *Labyrinth* puzzle

2) There are two doors: one leads you to the castle and one to certain death. Each door has a guard: one always lies and one always tells the truth. You can only ask one question to one of them. Which question you would ask to figure out which door to open?

-Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

-If guard A says S that means $\mathcal{A}$ = S.

-What else do we know? That $\mathcal{B} = \mathcal{A}$ is false. How to use that?

-Searching how to use the associativity of equality we try:

$(\mathcal{B} = \mathcal{A}) = S \qquad\qquad \mathcal{B} = (\mathcal{A} = S)$

-Therefore if we ask to B: what would A say if we ask him if the left door leads to the castle? We will know that the answer is a lie. That is, if the answer is yes then we should take the right door and if the answer is no we should take the left door.

Worksheet 5 - Knights and liars

3) The world is divided into two species, vampires and werewolves. Vampires always tell the truth when talking about a vampire, but always lie when talking about a werewolf. Werewolves always tell the truth when talking about a werewolf, but always lie when talking about a vampire. A says "B is a werewolf." Explain why A must be a werewolf.

Algebraic way:

-Suppose $\mathcal{A}$ means "A is a werewolf" and $\mathcal{B}$ means "B is a werewolf".

-If A says S about B that means $(\mathcal{A} = \mathcal{B}) = \text{S}$.

-A said "B is a werewolf", that is A said $\mathcal{B}$, therefore

$$(\mathcal{A} = \mathcal{B}) = \mathcal{B}$$

$$\mathcal{A} = (\mathcal{B} = \mathcal{B}) \qquad \text{(associativity of} = \text{)}$$

$$\mathcal{A} = \text{True} \qquad (\mathcal{B} = \mathcal{B} \text{ is True})$$

$\therefore$ A is a werewolf.

Worksheet 5 - Knights and liars

3) The world is divided into two species, vampires and werewolves. Vampires always tell the truth when talking about a vampire, but always lie when talking about a werewolf. Werewolves always tell the truth when talking about a werewolf, but always lie when talking about a vampire. A says "B is a werewolf." Explain why A must be a werewolf.

Truth table way: $\mathcal{A}$ means "A is a werewolf" and $\mathcal{B}$ "B is a werewolf".

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} = \mathcal{B}$	**Could A say $\mathcal{B}$?**
T	T	T	Yes
T	F	F	Yes
F	T	F	No
F	F	T	No

$\therefore$ A is a werewolf and we can't say anything about B.

This was part of the 2015 Oxford Maths Admission Test.

Worksheet 5 - Knights and liars

4) In the Knights and Liars Island you meet two natives (not tourists), A and B. What single question should you ask A to determine whether B is a knight?

Hint: Call Q the question that you want to ask, Q being a yes/no question.

Algebraic way:

-Suppose $\mathcal{A}$ means “A is a knight” and $\mathcal{B}$ means “B is a knight”.

-We know that if we ask Q to A the reply will be $Q = \mathcal{A}$.

-We want $(Q = \mathcal{A}) = \mathcal{B}$.

-By associativity of equality, that is $Q = (\mathcal{A} = \mathcal{B})$.

-Therefore we should ask A: are you and B of the same type?

Models and interpretations

Everything depends on the lens through which you see the world.

In science the lenses that we use are called models.

In history they are called interpretations.

What I am going to tell you about Socrates and Aristotle, two Greek philosophers from around 24 centuries ago, is just an interpretation.

The School of Athens

Raphael, 1509-1510, Vatican City..

Logic

How to use $p \Rightarrow q$ in practice?

Aristotle's way (forward): If we know p then we conclude q.

But how do we know p?

Find r such that $r \Rightarrow p$, find s such that $s \Rightarrow r$,...

That leads to what is called infinite regress.

How to solve that?

Aristotle's first principles	Axioms
Descartes's "I think therefore I am"	Dogmas

Aristotle wrote a lot of books, including physics, biology, zoology, metaphysics, ethics, aesthetics, poetry, theater, music, rhetoric, linguistics, politics, government and, importantly, the earliest formal study of logic.

We could interpret that he was very sure.

Plato and Aristotle

Aristotle was a student of Plato and Plato was a student of Socrates. Some people called them the Big Three in Greek philosophy.

Aristotle

Aristotle's influence was massive.

During centuries what he told was considered the truth. Thomas Aquinas and other Middle Ages thinkers used to refer to Aristotle as "The Philosopher".

In the 1940s Bertrand Russell wrote:
"Ever since the beginning of the 17th century, almost every serious intellectual advance has had to begin with an attack on some Aristotelian doctrine; in logic, this is still true at the present day."

Logic

In which other way we can use $p \Rightarrow q$ in practice?

Socrates way (backward): $\neg q \Rightarrow \neg p$

If someone states p and you find that p leads to a contradiction then you conclude that p is false.

A famous quote attributed to him is:

I know one thing: that I know nothing.

Consistent with that, Socrates did not write any book!

He became famous thanks to books written by some of his students, particularly Plato's *Dialogues*. His death made Socrates a martyr of philosophy.

Socrates

Socrates

In this interpretation, Socrates’s humble approach is closer to modern science: we have models, theories and we test them but it’s very hard, some may say impossible, to be 100% sure.

Similarly, in history we have interpretations but we are not sure. Remember, what I am telling you about Socrates and Aristotle is just an interpretation.

Here is a short story that, I think, shows that we can’t be 100% sure of knowing something.

Historical links

Socrates

Plato

Aristotle

Alexander the Great

Many Alexandrias, Library of Alexandria

Euclid

Euclid

Euclid's *Elements*

Euclid's *Elements* (published around 300 BC) was the first book that used the axiomatic method. Starting from few self-evident statements, called axioms, he proved, by logical deduction, a long list of theorems.

Euclid's *Elements* was a geometry textbook for more than 2000 years. Arguably the *Elements* is the most influential book in the history of mathematics.

The influence of its structure went beyond mathematics:

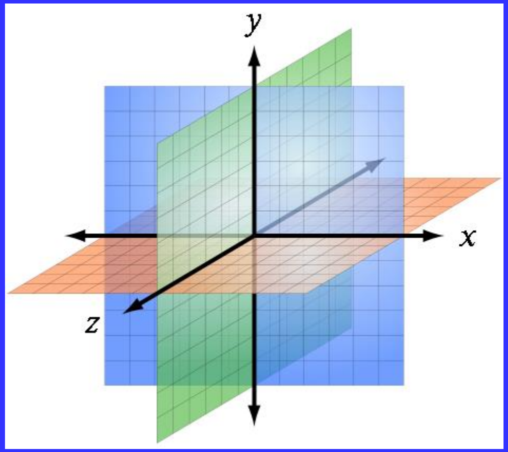
We hold these truths to be self-evident,…
that all men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty and the pursuit of Happiness.

United States Declaration of Independence

Euclid's *Elements*

For more than 2000 years people thought that the axioms in Euclid's *Elements* were true, that they corresponded to reality and that, therefore, all the theorems derived from them were true in the real world.

We used to think that the universe is a 3D Euclidean space:



Particularly, in Newton's *Principia* the universe is like that.

Non-Euclidean geometries

During more than 2000 years, mathematicians tried unsuccessfully to prove the 5th postulate of Euclid's *Elements* based on the first four.

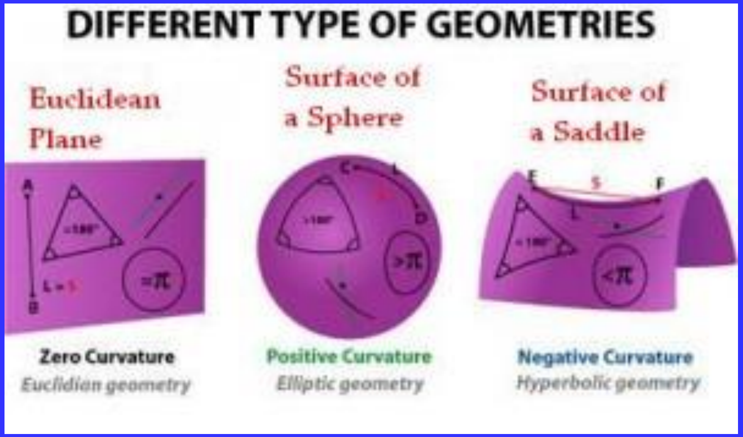
Finally in the 19th century Gauss, Bolyai, Lobachevsky and Riemmann discovered non-Euclidean geometries.

In a plane, given a line and a point not in it, how many parallels to that line pass through that point?

Euclidean geometry: 1

Elliptical geometry: 0

Hyperbolic geometry: Infinite



Non-Euclidean geometries

It was a big surprise in mathematics to find other axiomatic systems for geometry, that is other geometries.

A bigger surprise, I think, is that spacetime in Einstein's theory of relativity's is usually described as a non-Euclidean geometry!

It was similar to discovering that the Earth is not flat!

So, we thought that we knew the geometry of the universe but we just had a model.

The modern way to understand Euclid's *Elements* (and all the other axiomatic systems) is a like a p $\Rightarrow$ q: if all the axioms are true then all the theorems are true.

Socrates

Benjamin Franklin says in his *Autobiography*:

I was charmed with the Socratic method, I adopted it, I dropt my abrupt contradiction and positive argumentation, and put on the humble inquirer and doubter...

never using …the words *certainly*, *undoubtedly*…

but rather say… it appears to me,… or i*f I am not mistaken*…

a positive and dogmatical manner in advancing your sentiments may provoke contradiction and prevent a candid attention.

That's all!

Please fill the feedback questionnaire

Knights and liars

4) In the Knights and Liars Island you meet two natives (not tourists), A and B. What single question should you ask A to determine whether B is a knight?

Hint: Call Q the question that you want to ask, Q being a yes/no question.

Check with a truth table:

$\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} = \mathcal{B}$	**What would A answer to "Are you and B of the same type ?"**
T	T	T	Yes
T	F	F	No
F	T	F	Yes
F	F	T	No

Worksheet 5 - Knights and liars

5) One inhabitant of the island, A, says:

If I am a knight, then B is a liar.

What are A and B?

Algebraic way:

-Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

-If A lives on the island and says S that means $\mathcal{A}$ = S.

Given the implication, in order to use the associativity of = we recall:

$$(p \Rightarrow q) = (p = (p \wedge q))$$

$$(p \Rightarrow q) = (q = (p \vee q))$$

A says $(\mathcal{A} \Rightarrow \neg\mathcal{B})$

$\mathcal{A} = (\mathcal{A} \Rightarrow \neg\mathcal{B})$

$\mathcal{A} = (\mathcal{A} = (\mathcal{A} \wedge \neg\mathcal{B}))$

$(\mathcal{A} = \mathcal{A}) = (\mathcal{A} \wedge \neg\mathcal{B})$ (assoc. of =)

True $= (\mathcal{A} \wedge \neg\mathcal{B})$ ($\mathcal{A} = \mathcal{A}$ is True)

$\therefore$ $\mathcal{A}$ = True and $\mathcal{B}$ = False

$\therefore$ A is a knight and B is a liar.

Portia's caskets

7) Portia had two caskets: gold and silver. Inside one of these caskets, Portia had put her portrait, and on each was an inscription. Each inscription could be either true or false. The Gold casket's inscription was "The portrait is in this casket" and the Silver one was "If the inscription on the gold casket is true, this inscription is false." Which casket contained the portrait? What can we deduce about the inscriptions?

ig = Gold casket's inscription is true

is = Silver casket's inscription is true

G = The gold casket has the portrait

Given the implication, in order to use the associativity of = we recall:

$$(p \Rightarrow q) = (p = (p \wedge q))$$

$$(p \Rightarrow q) = (q = (p \vee q))$$

$ig = G$ (1)

$is = (ig \Rightarrow \neg is)$ (2)

$is = (\neg is = (ig \vee \neg is))$

$(is = \neg is) = (ig \vee \neg is)$ (assoc. of =)

$False = (ig \vee \neg is)$

$True = \neg (ig \vee \neg is)$ (negate both sides)

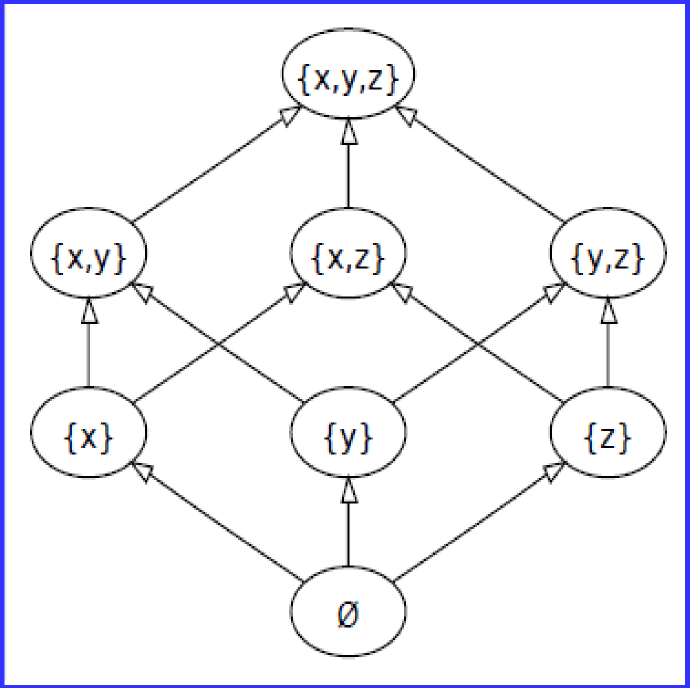
$True = (\neg ig \wedge is)$ (De Morgan)

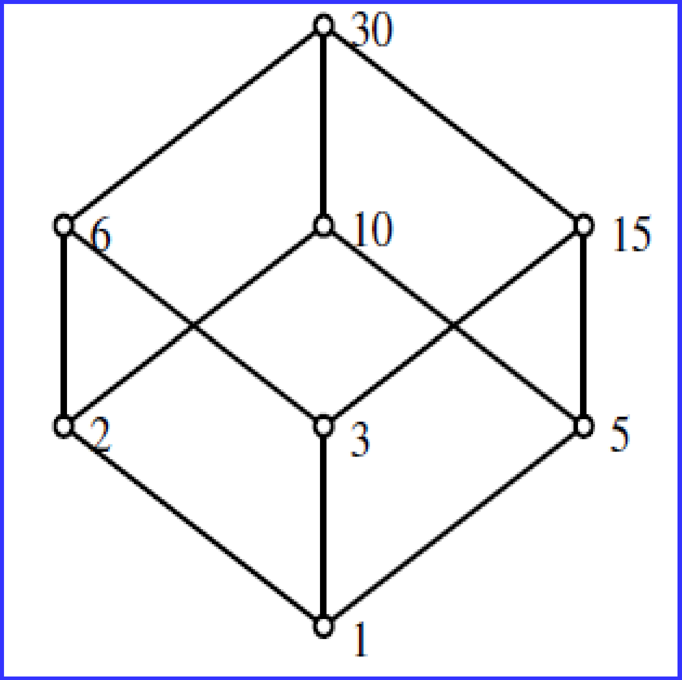
$\therefore$ ig = False and is = True

$\therefore$ G = False (sub. in (1))

Other Boolean algebras

The concept of Boolean algebra is actually more general than T, F, $\vee$, $\wedge$, $\neg$. Other Boolean algebras are:





A set, its subsets, union, intersection, complement.

A square-free natural number N, its divisors, lcm, hcf, N/d.

Other Boolean algebras

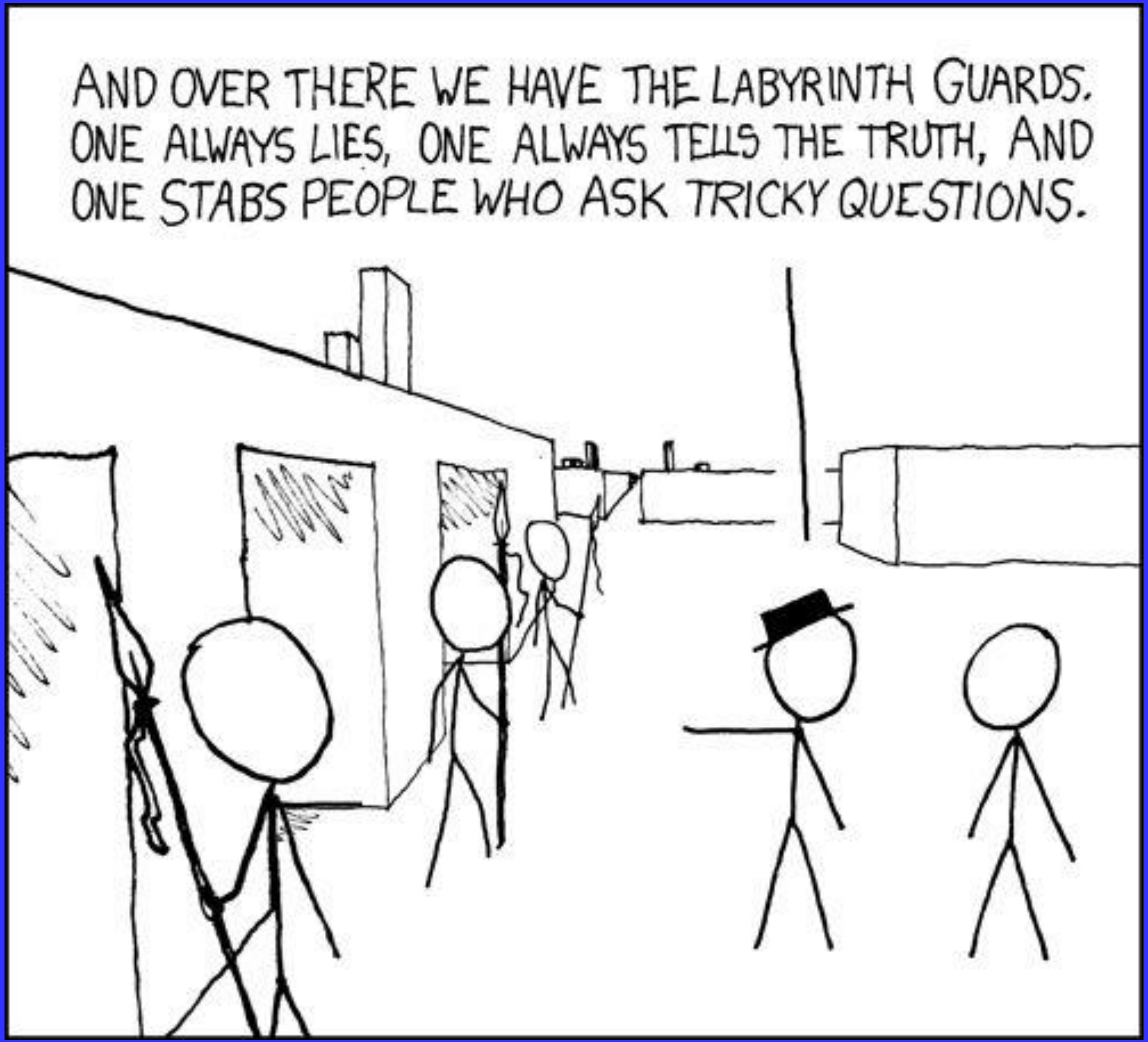
Abstract Boolean algebra

{True,False}

Subsets of a set

Divisors of a natural number

Knights and liars



https://www.xkcd.com/246/

https://www.explainxkcd.com/wiki/index.php/246:_Labyrinth_Puzzle

Socrates

Ancient Greek Philosophy video