Logic
Worksheet 1 - Reasoning

1) Do you agree with the following arguments?

   a) Pam promised she would attend the school assembly or send a substitute. She is not coming to the assembly, so we are expecting a substitute.

   b) If there are delays in the Circle line then Jenny is late at work. Today she has arrived late therefore there are delays in the Circle line.

   c) If there are delays in the Circle line then Jenny is late at work. Today she has arrived on time therefore there are no delays in the Circle line.

You don't have to finish these questions.

There is an island whose inhabitants are quite unusual. Some of them always tell the truth while others always lie. Those that always tell the truth are called knights.

2) While visiting the Knights and Liars Island, you meet a boy who tells you that he is a liar. Does he live on the island or is he a tourist?

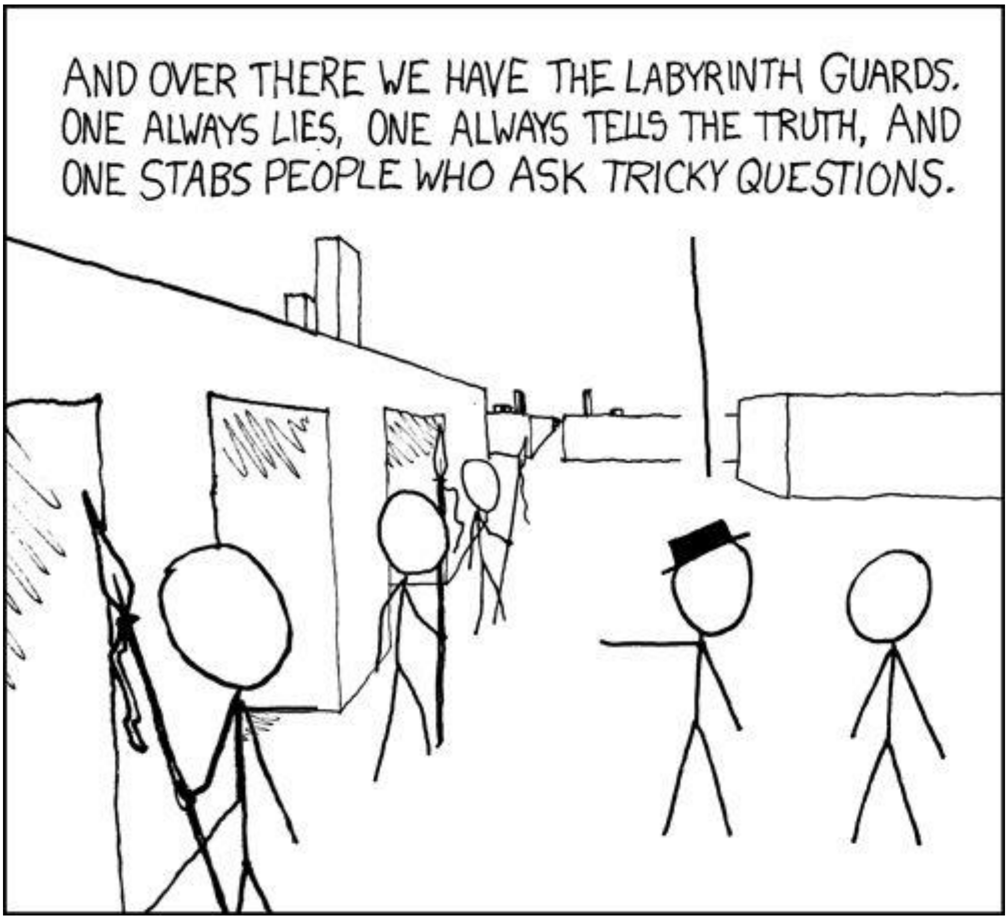
PLEASE TURN OVER

3) In the Knights and Liars Island you meet two natives (not tourists), A and B. What single question should you ask A to determine whether B is a knight?

4) In an abridged version of Shakespeare's Merchant of Venice, Portia had two caskets: gold and silver. Inside one of these caskets, Portia had put her portrait, and on each was an inscription. Portia explained to her suitor that each inscription could be either true or false but, on the basis of the inscriptions, he was to choose the casket containing the portrait. If he succeeded, he could marry her. The inscriptions were:

-Gold: The portrait is in this casket.
-Silver: If the inscription on the gold casket is true, this inscription is false.

Which casket contained the portrait? What can we deduce about the inscriptions?



https://www.xkcd.com/246/
https://www.explainxkcd.com/wiki/index.php/246:_Labyrinth_Puzzle

Logic
Worksheet 2 – Basic operations

Remember:

Logic	Sets
	$x \in P$ means x is a member of P (x is in P)
$\neg p$ means not p	P^c means the complement of P
$p \vee q$ means p or q	$P \cup Q$ means the union of P and Q
$p \wedge q$ means p and q	$P \cap Q$ means the intersection of P and Q

It's not a coincidence that $\vee$ and $\cup$ point down while $\wedge$ and $\cap$ point up.

p	$\neg p$
T	F
F	T

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

1) Do you agree with the following argument?
   - -I am at home or I am in the city.
   - -I am at home.
   - -Therefore, I am not in the city.

2) What about this one?
   - -I cannot be both at home and in the city.
   - -I am not at home.
   - -Therefore, I am in the city.

You don't have to finish these questions.

3) Could you draw lines pairing the expressions that are equal?

$p \vee q$
$p \wedge (q \vee r)$
$p \wedge q$
$\neg(p \vee q)$
$p \wedge (q \wedge r)$
$p \vee (q \vee r)$
$\neg(p \wedge q)$
$p \vee (q \wedge r)$

$(p \vee q) \wedge (p \vee r)$
$q \wedge p$
$(p \vee q) \vee r$
$q \vee p$
$(p \wedge q) \wedge r$
$\neg p \vee \neg q$
$(p \wedge q) \vee (p \wedge r)$
$\neg p \wedge \neg q$

Hint: Think about the equivalent set operations and Venn diagrams.

PLEASE TURN OVER

These questions are taken from United Kingdom Mathematics Trust (UKMT) Mathematical Challenges.

4) The four statements in the box refer to a mother and her four daughters. One statement is true, three statements are false.

Alice is the mother.
Carol and Ella are both daughters.
Beth is the mother.
One of Alice, Diane or Ella is the mother.

Who is the mother?

5) How many of the statements in the box are true?

None of these statements is true.
Exactly one of these statements is true.
Exactly two of these statements are true.
All of these statements are true.

Logic
Worksheet 3 – Truth tables

Remember:

How to prove that $(p \vee \neg p) = T$?

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

How to prove that $(p \wedge T) = p$?

p	T	$p \wedge T$
T	T	T
F	T	F

How to prove that $(p \wedge F) = F$?

p	F	$p \wedge F$
T	F	F
F	F	F

How to prove that $(p \vee T) = T$?

p	T	$p \vee T$
T	T	T
F	T	T

How to prove that $\neg(p \vee q) = \neg p \wedge \neg q$?

p	q	$p \vee q$	$\neg (p \vee q)$
T	T	T	**F**
T	F	T	**F**
F	T	T	**F**
F	F	F	**T**

$\neg p$	$\neg q$	$\neg p \wedge \neg q$
F	F	**F**
F	T	**F**
T	F	**F**
T	T	**T**

1) Using a truth table prove that $(p \wedge \neg p) =$ False. This is called the "no contradiction" law.

2) Using a truth table prove that $(p \vee \text{False}) = p$.

3) Using a truth table prove that $\neg (p \wedge q) = \neg p \vee \neg q$. This is a De Morgan's law.

PLEASE TURN OVER

You don't have to finish these questions.

4) Using a truth table prove that $p \wedge (q \wedge r) = (p \wedge q) \wedge r$. This is called the associativity of $\wedge$.

5) Using a truth table prove that $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$. This is called the distributivity of $\vee$ over $\wedge$.

Logic
Worksheet 4 - Implication

Remember:

Logic	Sets
$p \Rightarrow q$ means p implies q (if p then q)	$P \subseteq Q$ means P is a subset of Q

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

1) Peter's mom said: "All champions are good at maths." Peter says: "I am good at maths. Therefore I am a champion!" Is his reasoning correct?

2) There are four different cards on a table; each one has a letter on one side and a natural number on the other one. The visible sides have the symbols A, B, 4, and 5. What is the minimum number of cards we must turn over to find out whether the following statement is true: "If an even number is written on one side of a card then a vowel is written on the other side"? Which cards we need to turn over?

A	B	4	5

3) Using a truth table prove that $(p \Rightarrow q) \neq (q \Rightarrow p)$. This proves that *"If there are delays in the Circle line then Jenny is late at work. Today she has arrived late therefore there are delays in the Circle line."* is incorrect.

PLEASE TURN OVER

4) Using a truth table prove that $(p \Rightarrow q) = (\neg p \vee q)$.

You don't have to finish these questions.

5) Using a truth table prove that $(p \Rightarrow q) = (q = (p \vee q))$.

6) Using a truth table prove that $\big((p \Rightarrow q) \wedge (q \Rightarrow r)\big) \Rightarrow (p \Rightarrow r)$.

Logic
Worksheet 5 – Knights and liars

Remember:

Logic	Sets
p = q means p equals q	P = Q means the set P is a equal to the set Q

p	q	p = q
T	T	T
T	F	F
F	T	F
F	F	T

Equality is associative: $((p = q) = r) = (p = (q = r))$

Another useful property is: (p = True) = p

Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

If A lives in the K & L island and says S that means $\mathcal{A}$ = S.

1) While visiting the Knights and Liars Island, you meet a boy who tells you that he is a liar. Does he live on the island or is he a tourist?

2) *Labyrinth* puzzle. There are two doors: one leads you to the castle and one to certain death. Each door has a guard: one always lies and one always tells the truth. You can only ask one question to one of them. Which question you would ask to figure out which door to open?

3) The world is divided into two species, vampires and werewolves. Vampires always tell the truth when talking about a vampire, but always lie when talking about a werewolf. Werewolves always tell the truth when talking about a werewolf, but always lie when talking about a vampire. A says "B is a werewolf." Explain why A must be a werewolf.

PLEASE TURN OVER

You don't have to finish these questions.

4) In the Knights and Liars Island you meet two natives (not tourists), A and B. What single question should you ask A to determine whether B is a knight?
Hint: Call Q the question that you want to ask, Q being a yes/no question.

5) One inhabitant of the island, A, says: If I am a knight, then B is a liar.
What are A and B?
Hint: Use a truth table or use one of the following two formulas:

$$(p \Rightarrow q) = (p = (p \wedge q))$$
$$(p \Rightarrow q) = (q = (p \vee q))$$

6) In the Knights and Liars Island you meet two natives (not tourists), A and B. What single question should you ask A to determine whether A and B are different types?
Hint: Call Q the question that you want to ask, Q being a yes/no question.

7) In an abridged version of Shakespeare's Merchant of Venice, Portia had two caskets: gold and silver. Inside one of these caskets, Portia had put her portrait, and on each was an inscription. Portia explained to her suitor that each inscription could be either true or false but, on the basis of the inscriptions, he was to choose the casket containing the portrait. If he succeeded, he could marry her. The inscriptions were:

-Gold: The portrait is in this casket.
-Silver: If the inscription on the gold casket is true, this inscription is false.

Which casket contained the portrait? What can we deduce about the inscriptions?
Hint: Use one of the following two formulas:

$$(p \Rightarrow q) = (p = (p \wedge q))$$
$$(p \Rightarrow q) = (q = (p \vee q))$$

Logic masterclass
Acknowledgements
and more things to look at

I would like to thank Professor Paulo Oliva from Queen Mary University of London for his suggestions to improve this masterclass.

-Raymond Smullyan, A Beginner's Guide to Mathematical Logic. Raymond Smullyan popularised the knights and liars problems in his 1978 book What Is the Name of This Book? He published A Beginner's Guide to Mathematical Logic at the age of 95!

-George Boole, The Laws of Thought. This is the original book that showed how classical logic could be treated with algebraic terminology, including equations.

-The switching circuit images are mostly taken from I. M. Yaglom's Unusual algebra, which is part of a great Russian collection: https://mirtitles.org/2011/06/02/little-mathematics-library/

-Regula falsi method: https://en.wikipedia.org/wiki/False_position_method#Arithmetic_and_algebra

-Knights and liars: https://en.wikipedia.org/wiki/Knights_and_Knaves

-Algebraic approach to the knights and liars puzzles:

> -http://www.cs.nott.ac.uk/~psarb2/G51MPC/slides/KnightsAndKnaves.pdf
> -http://joaoff.com/wp-content/uploads/2011/06/thesis-a4-colour.pdf (page 192 onwards, several Knights and liars problems were taken from here).
> -Roland Backhouse, Algorithmic Problem Solving. Chapter 5.

-The Oxford Maths Admissions Test (MAT) usually includes a logic question: https://www.maths.ox.ac.uk/study-here/undergraduate-study/maths-admissions-test

-Once you master the knights and liars puzzles you could try "The Hardest Logic Puzzle Ever": https://en.wikipedia.org/wiki/The_Hardest_Logic_Puzzle_Ever .

-My interpretation about Aristotle and Socrates was inspired by Karl Popper's The open society and its enemies and by http://www.friesian.com/foundatn.htm .

-Raphael's The School of Athens: https://www.khanacademy.org/humanities/renaissance-reformation/high-ren-florence-rome/high-renaissance1/v/raphael-school-of-athens

-More about Socrates, Plato and Aristotle:

> -Bertrand Russell, A History of Western Philosophy. The chapter on Pythagoras has comments about the influence of geometry on the history of ideas.
> -http://www.dummies.com/education/philosophy/socrates-plato-and-aristotle-the-big-three-in-greek-philosophy/
> -http://ed.ted.com/on/tb7gSI6b#review

I hope you enjoyed the masterclass!
Gustavo Lau
You can send me questions and comments to gustavolau@gmail.com

Answers

Worksheet 1 - Reasoning

1) Do you agree with the following arguments?

   a) Pam promised she would attend the school assembly or send a substitute. She is not coming to the assembly, so we are expecting a substitute.

      Yes (assuming that we trust Pam). She stated p or q (where p = 'she would attend the school assembly' and q = 'she would send a substitute'). We are told that p is false so the only way that (p or q) can be true is if q is true.

   b) If there are delays in the Circle line then Jenny is late at work. Today she has arrived late therefore there are delays in the Circle line.

      No. Jenny could be late for a different reason than the Circle line having delays. For example, she could have overslept.

   c) If there are delays in the Circle line then Jenny is late at work. Today she has arrived on time therefore there are no delays in the Circle line.

      Yes. If there were delays in the Circle line we would have a contradiction: she would be late but she has arrived on time.

There is an island whose inhabitants are quite unusual. Some of them always tell the truth while others always lie. Those that always tell the truth are called knights.

2) While visiting the Knights and Liars Island, you meet a boy who tells you that he is a liar. Does he live on the island or is he a tourist?
   See answer 5.1.

3) In the Knights and Liars Island you meet two natives (not tourists), A and B. What single question should you ask A to determine whether B is a knight?
   See answer 5.4.

4) In an abridged version of Shakespeare's Merchant of Venice, …
   See answer 5.7.

Worksheet 2 - Basic operations

1) Do you agree with the following argument?
   -I am at home or I am in the city.
   -I am at home.
   -Therefore, I am not in the city.

   No, as I could be in both the city and at home. Using the truth table:

I am at home	I am in the city	I am at home or I am in the city
T	T	T
T	F	T
~~F~~	~~T~~	~~T~~
~~F~~	~~F~~	~~F~~

   The first statement discards the 4th row and the second one discards the 3rd row. In the two rows left "I am in the city" could be true or false.

2) What about this one?
   -I cannot be both at home and in the city.
   -I am not at home.
   -Therefore, I am in the city.

   No, as I could be not at home and not in the city. Using the truth table:

I am at home	I am in the city	I am at home and I am in the city
~~T~~	~~T~~	~~T~~
~~T~~	~~F~~	~~F~~
F	T	F
F	F	F

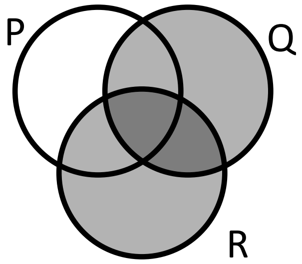
   The first statement discards the 1st row and the second one discards the 2nd row. In the two rows left "I am in the city" could be true or false.

3) Could you draw lines pairing the expressions that are equal?

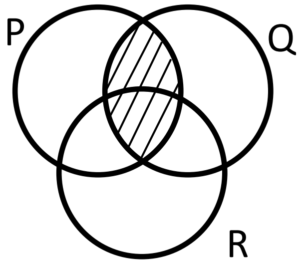
One way of doing this is using Venn diagrams for the analogous set operations.

An example: $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

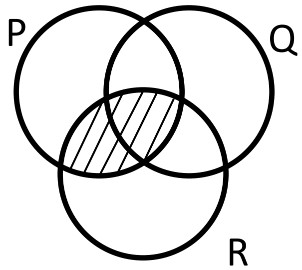
$Q \cup R$:



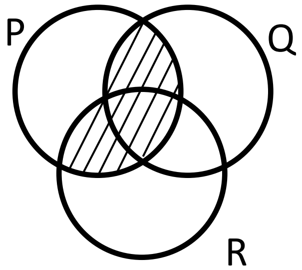
$P \cap Q$:



$P \cap R$:



$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$:



The correct pairings are:

$p \vee q$	$(p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r)$	$q \wedge p$
$p \wedge q$	$(p \vee q) \vee r$
$\neg(p \vee q)$	$q \vee p$
$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r$
$p \vee (q \vee r)$	$\neg p \vee \neg q$
$\neg(p \wedge q)$	$(p \wedge q) \vee (p \wedge r)$
$p \vee (q \wedge r)$	$\neg p \wedge \neg q$

The names of the properties are:

			Name	**Similar to**
$p \vee q$	=	$q \vee p$	Commutativity of $\vee$	a+b = b+a
$p \wedge q$	=	$q \wedge p$	Commutativity of $\wedge$	ab = ba
$p \vee (q \vee r)$	=	$(p \vee q) \vee r$	Associativity of $\vee$	a+(b+c) = (a+b)+c
$p \wedge (q \wedge r)$	=	$(p \wedge q) \wedge r$	Associativity of $\wedge$	a(bc) = (ab)c
$p \vee (q \wedge r)$	=	$(p \vee q) \wedge (p \vee r)$	Distributivity of $\vee$ over $\wedge$	
$p \wedge (q \vee r)$	=	$(p \wedge q) \vee (p \wedge r)$	Distributivity of $\wedge$ over $\vee$	a(b+c) = ab + ac
$\neg(p \vee q)$	=	$\neg p \wedge \neg q$	De Morgan's law	
$\neg(p \wedge q)$	=	$\neg p \vee \neg q$	De Morgan's law	

4) The four statements in the box refer to a mother and her four daughters. One statement is true, three statements are false.

Alice is the mother.
Carol and Ella are both daughters.
Beth is the mother.
One of Alice, Diane or Ella is the mother.

Who is the mother?

We analyse the 5 possibilities for the mother:

If Alice were the mother then the first and fourth statements would be true.

If Beth were the mother then the second and third statements would be true.

If Carol were the mother then all the statements would be false.

If Diane were the mother the second and fourth statements would be true.

If Ella were the mother then the fourth statement would be true and the rest would be false.

Since we are told that one statement is true and three statements are false we conclude that Ella is the mother.

5) How many of the statements in the box are true?

None of these statements is true. Exactly one of these statements is true. Exactly two of these statements are true. All of these statements are true.

Only one statement is true, the second one. Any other statement leads to a contradiction.

Worksheet 3 - Truth tables

1) Using a truth table prove that $(p \wedge \neg p) = \text{False}$. This is called the "no contradiction" law.

p	$\neg$ p	p $\wedge$ $\neg$p
T	F	F
F	T	F

2) Using a truth table prove that $(p \vee \text{False}) = p$.

p	F	p $\vee$ F
T	F	**T**
F	F	**F**

The identity, or neutral, element of an operation leaves other elements unchanged when combined with them. For example, 0 is the identity element of the sum because, for every number a, a+0 = a. Therefore $(p \vee \text{False}) = p$ means that False is the identity element of $\vee$.

3) Using a truth table prove that $\neg (p \wedge q) = \neg p \vee \neg q$. This is a De Morgan's law.

p	q	p $\wedge$ q	$\neg$ (p $\wedge$ q)
T	T	T	**F**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

$\neg$p	$\neg$q	$\neg$p $\vee$ $\neg$q
F	F	**F**
F	T	**T**
T	F	**T**
T	T	**T**

5) Using a truth table prove that $p \wedge (q \wedge r) = (p \wedge q) \wedge r$. This is called the associativity of $\wedge$.

p	q	r	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	**T**
T	T	F	F	**F**
T	F	T	F	**F**
T	F	F	F	**F**
F	T	T	T	**F**
F	T	F	F	**F**
F	F	T	F	**F**
F	F	F	F	**F**

$p \wedge q$	$(p \wedge q) \wedge r$
T	**T**
T	**F**
F	**F**
F	**F**
F	**F**
F	**F**
F	**F**
F	**F**

6) Using a truth table prove that $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$. This is called the distributivity of $\vee$ over $\wedge$.

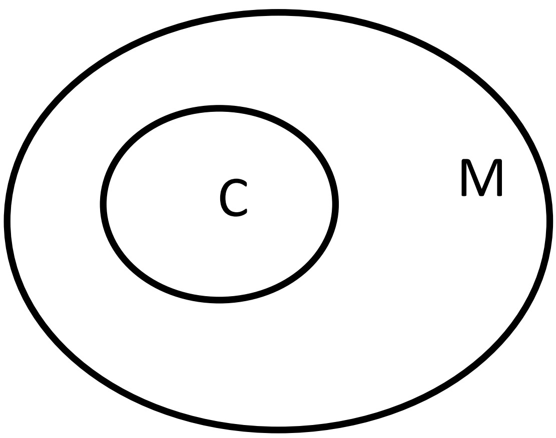
p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	**T**
T	T	F	F	**T**
T	F	T	F	**T**
T	F	F	F	**T**
F	T	T	T	**T**
F	T	F	F	**F**
F	F	T	F	**F**
F	F	F	F	**F**

$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	**T**
T	T	**T**
T	T	**T**
T	T	**T**
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**F**

Worksheet 4 - Implication

1) Peter's mom said: "All champions are good at maths." Peter says: "I am good at maths. Therefore I am a champion!" Is his reasoning correct?

No. What Peter's mom says is equivalent to say that the set of champions is a subset of the people that are good at maths but not that those two sets are equal. So, according to what his mom said, Peter could be good at maths and not a champion.



C *= set of champions*

M *= set of people that are good at maths*

2) There are four different cards on a table; each one has a letter on one side and a natural number on the other one. The visible sides have the symbols A, B, 4, and 5. What is the minimum number of cards we must turn over to find out whether the following statement is true: "If an even number is written on one side of a card then a vowel is written on the other side"?" Which cards we need to turn over?

A	B	4	5

Two. The statement is "Even implies Vowel". Looking at the table below it is clear that we only need to turn over the 4 and the B, turning over the other two would not give us any information.

Even	Vowel	Even $\Rightarrow$ Vowel
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

3) Using a truth table prove that $(p \Rightarrow q) \neq (q \Rightarrow p)$. This proves that "If there are delays in the Circle line then Jenny is late at work. Today she has arrived late therefore there are delays in the Circle line." is incorrect.

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	**T**	**T**
T	F	**F**	**T**
F	T	**T**	**F**
F	F	**T**	**T**

4) Using a truth table prove that $(p \Rightarrow q) = (\neg p \vee q)$.

p	q	$p \Rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

$\neg p$	$\neg p \vee q$
F	**T**
T	**F**
T	**T**
T	**T**

5) Using a truth table prove that $(p \Rightarrow q) = (q = (p \vee q))$.

p	q	$p \Rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

$p \vee q$	$q = (p \vee q)$
T	**T**
T	**F**
T	**T**
F	**T**

7) Using a truth table prove that $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

Given that the last column has only T's, $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$ is always true. This is called the transitivity of the implication.

Worksheet 5 - Knights and liars

1) While visiting the Knights and Liars Island, you meet a boy who tells you that he is a liar. Does he live on the island or is he a tourist?

Case analysis way:

If the boy is a knight then he would be lying when he says that he is a liar. That is a contradiction.
If he is a liar then he would be telling the truth when he says that he is a liar. That is a contradiction.
Therefore he can't live on the island.
Therefore he is he is a tourist.

Algebraic way:

-Let's call the boy B.

-Suppose $\mathcal{B}$ means "B is a knight".

-If B lives in the island and says S that means $\mathcal{B}$ = S.

-B said (B = False). If B were to live in the island we would have

$\mathcal{B} = (\mathcal{B} = \text{False})$

$(\mathcal{B} = \mathcal{B}) = \text{False}$ (associativity of =)

True = False   ($\mathcal{B} = \mathcal{B}$ is True)

A contradiction, therefore B is a tourist.

2) Labyrinth puzzle. There are two doors: one leads you to the castle and one to certain death. Each door has a guard: one always lies and one always tells the truth. You can only ask one question to one of them. Which question you would ask to figure out which door to open?

Algebraic way:

-Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

-If guard A says S that means $\mathcal{A}$ = S.

-What else do we know?

That $\mathcal{B} = \mathcal{A}$ is false. How to use that?

-Searching how to use the associativity of equality we try:

$(\mathcal{B} = \mathcal{A}) = S$

$\mathcal{B} = (\mathcal{A} = S)$

-Therefore if we ask to B: what would A say if we ask him if the left door leads to the castle? We will know that the answer is a lie. That is, if the answer is yes then we should take the right door and if the answer is no we should take the left door.

Check with case analysis:

Let's call Q the question "what would A say if we ask him if the left door leads to the castle?" We have four cases:

a) The left door leads to the castle and A always lies. Then if we were to ask A if the left door leads to the castle he would say "No". In this case B always tells the truth and to question Q he would reply "No".
b) The left door leads to the castle and A always tells the truth. Then if we were to ask A if the left door leads to the castle he would say "Yes". In this case B always lies and to question Q he would reply "No".
c) The left door leads certain death and A always lies. If we were to ask A if the left door leads to the castle he would say "Yes". In this case B always tells the truth and to question Q he would reply "Yes".
d) The left door leads certain death and A always tells the truth. Then if we were to ask A if the left door leads to the castle he would say "No". In this case B always lies and to question Q he would reply "Yes".

Therefore if we pose question Q and B replies "No" we should open the left door and if he replies "Yes" we should open the right one.

3) The world is divided into two species, vampires and werewolves. Vampires always tell the truth when talking about a vampire, but always lie when talking about a werewolf. Werewolves always tell the truth when talking about a werewolf, but always lie when talking about a vampire. A says "B is a werewolf." Explain why A must be a werewolf.

Algebraic way:

-Suppose $\mathcal{A}$ means "A is a werewolf" and $\mathcal{B}$ means "B is a werewolf".

-If A says S about B that means ($\mathcal{A} = \mathcal{B}$) = S.

-A said "B is a werewolf", that is A said $\mathcal{B}$, therefore

$(\mathcal{A} = \mathcal{B}) = \mathcal{B}$

$\mathcal{A} = (\mathcal{B} = \mathcal{B})$ (associativity of =)

$\mathcal{A}$ = True ($\mathcal{B} = \mathcal{B}$ is True)

Therefore A is a werewolf.

Truth table way:

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} = \mathcal{B}$	Could A say $\mathcal{B}$?
T	T	T	**Yes**
T	F	F	**Yes**
F	T	T	**No**
F	F	T	**No**

Therefore A is a werewolf and we can't say anything about B.

4) In the Knights and Liars Island you meet two natives (not tourists), A and B. What single question should you ask A to determine whether B is a knight?

-Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

-Call Q the question that you want to ask.

-We know that if we ask Q to A the reply will be Q = $\mathcal{A}$.

-So we want (Q = $\mathcal{A}$) = $\mathcal{B}$.

-By associativity of equality, that is Q = ($\mathcal{A} = \mathcal{B}$).

-Therefore we should ask A: are you and B of the same type?

Let's check with a truth table:

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A} = \mathcal{B}$	What would A answer to: are A and B of the same type?
T	T	T	Yes
T	F	F	No
F	T	F	Yes
F	F	T	No

So the answer will be yes if and only if B is a Knight.

5) One inhabitant of the island, A, says: If I am a knight, then B is a liar. What are A and B?

Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

Truth table way:

$\mathcal{A}$	$\mathcal{B}$	$\neg\mathcal{B}$	$\mathcal{A} \Rightarrow \neg\mathcal{B}$	Could A say $\mathcal{A} \Rightarrow \neg\mathcal{B}$?
T	T	F	F	**No**
T	F	T	T	**Yes**
F	T	F	T	**No**
F	F	T	T	**No**

Therefore A is a knight and B is a liar.

Algebraic way:

-If A lives in the island and says S that means $\mathcal{A}$ = S.

-A says $(\mathcal{A} \Rightarrow \neg\mathcal{B})$. Therefore

$$\mathcal{A} = (\mathcal{A} \Rightarrow \neg\mathcal{B}) \qquad (1)$$

In order to use the associativity of equality the formula that helps is

$$(p \Rightarrow q) = (p = (p \wedge q))$$

Applying that to the right side of (1) we get

$$\mathcal{A} = (\mathcal{A} = (\mathcal{A} \wedge \neg\mathcal{B}))$$

$$(\mathcal{A} = \mathcal{A}) = (\mathcal{A} \wedge \neg\mathcal{B}) \qquad \text{(associativity of =)}$$

$$\text{True} = (\mathcal{A} \wedge \neg\mathcal{B}) \qquad (\mathcal{A} = \mathcal{A} \text{ is True})$$

$$\mathcal{A} = \text{True and } \neg\mathcal{B} = \text{True}$$

$$\mathcal{A} = \text{True and } \mathcal{B} = \text{False}$$

Therefore A is a knight and B is a liar.

6) In the Knights and Liars Island you meet two natives (not tourists), A and B. What single question should you ask A to determine whether A and B are different types?

Suppose $\mathcal{A}$ means "A is a knight" and $\mathcal{B}$ means "B is a knight".

Call Q the question that you want to ask. We know that if we ask Q to A the reply will be Q = $\mathcal{A}$. So we want

$(Q = \mathcal{A}) = (\mathcal{A} = \neg\mathcal{B})$

$((Q = \mathcal{A}) = \mathcal{A}) = \neg\mathcal{B}$ (associativity of =)

$(Q = (\mathcal{A} = \mathcal{A})) = \neg\mathcal{B}$ (associativity of =)

$(Q = \text{True}) = \neg\mathcal{B}$ ($\mathcal{A} = \mathcal{A}$ is always true)

$Q = \neg\mathcal{B}$ ((Q=True) is equal to Q)

Therefore we should ask A: is B a liar?

Let's check with a truth table:

$\mathcal{A}$	$\mathcal{B}$	What would A answer to: is B a liar?
T	T	No
T	F	Yes
F	T	Yes
F	F	No

So the answer will be yes if and only if A and B are of different types.

7) In an abridged version of Shakespeare's Merchant of Venice, Portia had two caskets: gold and silver. Inside one of these caskets, Portia had put her portrait, and on each was an inscription. Portia explained to her suitor that each inscription could be either true or false but, on the basis of the inscriptions, he was to choose the casket containing the portrait. If he succeeded, he could marry her. The inscriptions were:

-Gold: The portrait is in this casket.
-Silver: If the inscription on the gold casket is true, this inscription is false.

Which casket contained the portrait? What can we deduce about the inscriptions?

Let's define the following Boolean variables:

G = The gold casket has the portrait

ig = The inscription in the gold casket is true

is = The inscription in the silver casket is true

Truth table way:

G	ig	is	G = ig	$\neg$is	ig $\Rightarrow$ $\neg$is	is = (ig $\Rightarrow$ $\neg$is)	Could it be?
T	T	T	T	F	F	F	No, because is = (ig $\Rightarrow$ $\neg$is) is F
T	T	F	T	T	T	F	No, because is = (ig $\Rightarrow$ $\neg$is) is F
T	F	T	F	F	T	T	No, because G = ig is F
T	F	F	F	T	T	F	No, because G = ig is F
F	T	T	F	F	F	F	No, because G = ig is F
F	T	F	F	T	T	F	No, because G = ig is F
F	F	T	T	F	T	T	Yes
F	F	F	T	T	T	F	No, because is = (ig $\Rightarrow$ $\neg$is) is F

Therefore G=F, ig=F and is=T, that is the portrait is in the silver casket, the inscription in the gold casket is false and the one in the silver one is true.

Algebraic way:

Similarly to the Knights and liars, when an inscription I says X we don't know if X is true or false, we only know that I = X. Therefore, the system of equations is:

$$ig = G \qquad (1)$$

$$is = (ig \Rightarrow \neg is) \qquad (2)$$

Let's work with equation (2). Given that one side of the equality is an implication, in order to use the associativity of equality we look at these equalities:

$$(p \Rightarrow q) = (p = (p \wedge q)) \qquad (3)$$

$$(p \Rightarrow q) = (q = (p \vee q)) \qquad (4)$$

Given that in (2) the left side of the equality is the negation of the right side of the implication we use (4) to convert (2) into:

$$is = (\neg is = (ig \vee \neg is))$$

Now we use the associativity of equality:

$$(is = \neg is) = (ig \vee \neg is)$$

Given that is = ¬is is False we get:

$$(ig \vee \neg is) = False$$

That is:

$$\neg\, (ig \vee \neg is)$$

By De Morgan, i.e., $\neg(ig \vee \neg is) = (\neg ig \wedge is)$,

$$is \wedge \neg ig$$

$$(is = True) \wedge (ig = False)$$

By (1) we get:

$$G = False$$

Therefore the portrait is in the silver casket, the inscription in the gold casket is false and the one in the silver one is true.